

# Goodness of Fit Tests

Documentation on `libcdhc.a`

and

A GRASS Tutorial on `s.normal`

James Darrell McCauley\*

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## Abstract

The methods used by the GRASS program `s.normal` are presented. These are various goodness of fit statistics for testing the null hypothesis of normality. Other additional tests found in `cdhc` a C programming library, are also documented (this document serves two puposes: a tutorial for the GRASS geographic information system and documentation for the library).

## 1 Introduction

This document is a programmer's manual for `cdhc`, a C programming library useful for testing whether a sample is normally, lognormally, or exponentially distributed. Prototypes for library functions<sup>1</sup> are given in the margins near corresponding mathematical explanations. Hence, it is also a user's guide for programs using `cdhc`. Readers should be equipped with at least one graduate course in probability and statistics. Much of the background and derivation/justification of each test has been omitted. A good text for more background information is *Goodness-of-Fit Techniques* by D'Agostino and Stephens [13] (see also references in text).

### 1.1 Hypothesis Testing

Before beginning the description of the tests, a few definitions should be given. The general framework for mosts tests is that the *null* hypothesis  $H_0$  is that a random variable  $x$  follows a particular distribution  $F(x)$ . Generally, the *alternative* hypothesis is that  $x$  does not follow  $F(x)$  (with no additional usable

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\*USDA National Needs Fellow, Department of Agricultural Engineering, Purdue University.  
Email: [mccauley@ecn.purdue.edu](mailto:mccauley@ecn.purdue.edu)

<sup>1</sup>Each function in the library returns a pointer to static double. The `cdhc` library was inspired by Johnson's STATLIB collection of FORTRAN routines for testing distribution assumptions [22]. Some functions in `cdhc` are loosely based on Johnson's work (they have been completely rewritten, reducing memory requirements and number of computations and fixing a few bugs). Others are based on algorithms found in *Applied Statistics*, *Technometrics*, and other related journals.

information; the Kotz Separate Families test in §8 is one exception). This may differ from the way that some have learned hypothesis testing in that some tests are set up to reject the null hypothesis in favor of the alternative.

A *simple* hypothesis implies that  $F(x)$  is completely specified, e.g.,  $x \sim N(0, 1)$ . A *composite* hypothesis means that one (or more) of the parameters of  $F(x)$  is not completely specified, e.g.,  $x \sim N(\mu, \sigma)$ . That is, the composite hypothesis may be:

$$H_0 : F(x) = F_0(x; \theta)$$

where  $\theta = [\theta_1, \dots, \theta_p]'$  is a  $p$  vector of *nuisance* parameters whose values are unknown and must be estimated from data.

## 1.2 Probability Plots

In addition to these analytical techniques, graphical methods are valuable supplements. The most important graphical technique is probability plotting. A *probability plot* is a plot of the cumulative distribution function  $F(x)$  on the vertical axis versus  $x$  on the horizontal axis. The vertical axis is scaled such that, if the data fit the assumed distribution, the resulting plot will lie on a straight line. Special plotting paper may be purchased to do these plots; however, most modern scientific plotting programs have this capability (e.g., `gnuplot`). Each test presented below should be used in conjunction with a probability plot.

## 1.3 Shape of Distributions

Through much of the literature are references to Johnson curves:  $S_U$  or  $S_B$  (see §2, page 3). These refer to a system of distributions introduced by Johnson [21] where a standard normal random variable  $Z$  is translated to  $(Z - \gamma)/\delta$  and transformed using  $T$ :

$$Y = T\left(\frac{Z - \gamma}{\delta}\right). \quad (1)$$

Three families in Johnson's [21] system are:

1. a family of bounded distributions, denoted by  $S_B$ , where:

$$Y = T\left(\frac{e^x}{1 + e^x}\right); \quad (2)$$

2. a family lognormal distributions where:

$$Y = T(e^x); \quad (3)$$

3. and a family of unbounded distributions, denoted by  $S_U$ , where:

$$Y = \sinh(x) = T(e^x - e^{-x}). \quad (4)$$

In the  $S_B$  and  $S_U$  families,  $\gamma$  and  $\delta$  govern the shape of the distribution. In the lognormal families,  $\delta$  governs the shape while  $\gamma$  is only a scaling factor [20]. Other approaches to exploring the shape of a distribution include  $g$ - and  $h$ -distributions [20] and Pearson curves (see Bowman [4]).

## 1.4 Miscellaneous

Many tests are presented here without mention of their relative merits. Users are advised to consult the cited literature to determine which test is appropriate for their situation. Sometimes a certain test will have more *power* than another; that is, a test may have a better ability to reject a model when the model is incorrect.

## 2 Moments: $b_2$ and $\sqrt{b_1}$

```
double*
omnibus_moments(x,n)
double *x;
int n;
Returns [ $\sqrt{b_1}, b_2$ ]'.
```

Let  $x_1, x_2, \dots, x_n$  be the  $n$  observations with mean:

$$m_1 = \frac{1}{n} \sum_{j=1}^n nx_j. \quad (5)$$

The central moments are defined as:

$$m_i = \frac{1}{n} \sum_{j=1}^n n(x_j - m_i)^i, \quad i = 2, 3, 4. \quad (6)$$

The sample skewness ( $\sqrt{b_1}$ ) and kurtosis ( $b_2$ ) are defined as:

$$\sqrt{b_1} = m_3/m_2^{3/2} = \sqrt{n} \left( \sum_{j=1}^n (x_i - \bar{x})^3 \right) / \left( \sum_{j=1}^n (x_i - \bar{x})^2 \right)^{3/2} \quad (7)$$

and

$$b_2 = m_4/m_2^2. \quad (8)$$

These are invariant under both origin and scale changes [4]. When a distribution is specified, these are denoted as  $\sqrt{\beta_1}$  and  $\beta_2$ .

For a standard normal,  $\sqrt{\beta_1} = 0$  and  $\beta_2 = 3$ . To use either or both of these statistics to test for departure from normality, these are sometimes transformed to their standardized to their normal equivalent deviates,  $X(\sqrt{b_1})$  and  $X(b_1)$ .

For  $X(\sqrt{b_1})$ , D'Agostino and Pearson [11] gave coefficients  $\delta$  and  $\lambda$  ( $n = 8$  to 1000) for:

$$X(\sqrt{b_1}) = \delta \sinh^{-1}(\sqrt{b_1}/\lambda) \quad (9)$$

that transforms  $\sqrt{b_1}$  to a standard normal using a Johnson  $S_U$  approximation (Table 5). An equivalent approximation [10] that avoids the use of tables is given by:

1. Compute  $\sqrt{b_1}$  from the sample data.

2. Compute:

$$Y = \sqrt{b_1} \left[ \frac{(n+1)(n+3)}{6(n-2)} \right]^{\frac{1}{2}}, \quad (10)$$

$$\beta_2 = \frac{3(n^2 + 27n - 70)(n+1)(n+3)}{(n-2)(n+5)(n+7)(n+9)}, \quad (11)$$

$$W^2 = \sqrt{2(\beta_2 - 1)} - 1, \quad (12)$$

$$\delta = 1/\sqrt{\log W}, \quad (13)$$

$$\text{and} \quad (14)$$

$$\alpha = \sqrt{2/(W^2 - 1)}. \quad (15)$$

3. Compute the standard normal variable:

$$Z = \delta \log \left[ Y/\alpha + \sqrt{(Y/\alpha)^2 + 1} \right]. \quad (16)$$

This procedure is applicable for  $n \geq 8$ .

---

*Example:* For the sample data given in Table 4 ( $n = 584$ ),  $\sqrt{b_1} = 0.2373$ . Suppose that we wish to test the hypothesis of normality:

$H_0: \sqrt{\beta_1} = 0$  (normality)

versus the two-sided alternative

$H_1: \sqrt{\beta_1} \neq 0$  (non-normality)

at a level of significance of 0.05. Following the procedure given above,  $Y = 2.3454$ ,  $\beta_2 = 3.0592$ ,  $W^2 = 1.0294$ ,  $\delta = 12.6132$ ,  $\alpha = 8.2522$ , and  $Z = 1.5367$ . At a 0.05 significance level for a two-sided test, we reject the null hypothesis of normality if  $|Z| \geq 1.96$ . In this instance, we cannot reject  $H_0$ .

---

The fourth standardized moment  $b_2$  may be used to test the normality hypothesis by the following procedure [2]:

1. Compute  $b_2$  from the sample data.

2. Compute the mean and variance of  $b_2$ :

$$E(b_2) = \frac{3(n-1)}{n+1} \quad (17)$$

and

$$Var(b_2) = \frac{24n(n-2)(n-3)}{(n+1)^2(n+3)(n+5)}. \quad (18)$$

3. Compute the standardized value of  $b_2$ :

$$y = \frac{b_2 - E(b_2)}{Var(b_2)}. \quad (19)$$

4. Compute the third standardized moment of  $b_2$ :

$$\sqrt{\beta_1(b_2)} = \frac{6(n^2 - 5n + 2)}{(n+7)(n+9)} \sqrt{\frac{6(n+3)(n+5)}{n(n-2)(n-3)}}. \quad (20)$$

5. Compute:

$$A = 6 + \frac{8}{\sqrt{\beta_1(b_2)}} \left[ \frac{2}{\sqrt{\beta_1(b_2)}} + \sqrt{1 + \frac{4}{\sqrt{\beta_1(b_2)}}} \right]. \quad (21)$$

6. Compute:

$$Z = \left( \left( 1 - \frac{2}{9A} \right) - \left[ \frac{1 - 2/A}{1 + y\sqrt{2/(A-4)}} \right]^{\frac{1}{3}} \right) / \sqrt{2/(9A)} \quad (22)$$

where  $Z$  is a standard normal variable with zero mean and variance of one.

*Example:* For the sample data given in Table 4 ( $n = 584$ ),  $b_2 = 1.9148$ . Suppose that we wish to test the hypothesis of normality:

$H_0: \beta_2 = 3$  (normality)

versus the one-sided alternative

$H_1: \beta_2 > 3$  (non-normality)

at a level of significance of 0.05. We would reject  $H_0$  if  $Z$  (eqn. 22) is larger than 1.645 (Table 1). Following the procedure given above,  $E(b_2) = 2.9897$ ,  $Var(b_2) = 0.0401$ ,  $y = -26.8366$ ,  $\sqrt{\beta_1(b_2)} = 0.0989$ ,  $A = 2163$ , and  $Z = -131.7$ . Therefore, we cannot reject  $H_0$ .

## 2.1 Omnibus Tests for Normality

## 3 Geary's Test of Normality

```
double*
geary_test(x,n)
double *x;
int n;
Returns [ $\sqrt{a}, y$ ]'.
```

Let  $x_1, x_2, \dots, x_n$  be the  $n$  observations. The ratio of the mean deviation to the standard deviation is given as:

$$a = \frac{1}{n\sqrt{m_2}} \sum_{j=1}^n |x_j - \bar{x}| \quad (23)$$

where  $\bar{x} = \sum_{i=1}^n x_i$  and  $m_2$  is defined by eqn. 6. This ratio can be transformed a standard normal [10] via

$$y = \frac{\sqrt{n}(a - 0.7979)}{0.2123}. \quad (24)$$

This test is valid for  $n \geq 41$ .

More generally, Geary [17] considered tests of the form

$$a(c) = \frac{1}{nm_2^{c/2}} \sum_{j=1}^n |x_j - \bar{x}|^c \text{ for } c \geq 1 \quad (25)$$

where  $a(1) = a$  of eqn. 23, and  $a(4) = b_2$  of eqn. 8.

D'Agostino and Rosman [12] conclude that Geary's  $a$  test has good power for symmetric alternatives and skewed alternatives with  $\beta_2 < 3$  when compared to other tests, though for symmetric alternatives,  $b_2$  (eqn. 8) can sometimes be more powerful and for skewed alternatives,  $W$  (eqn 70) or  $W'$  (eqn 73) usually dominate  $a$ . The Geary test (eqns. 23-24) is seldom used today—D'Agostino [10] include it in his summary work because it is of “historical interest.”

*Example:* For the sample data given in Table 4 ( $n = 584$ ),  $a = 0.8823$ . Suppose that we wish to test the hypothesis of normality:

$H_0$ : normality

versus the two-sided alternative

$H_1$ : non-normality

at a level of significance of 0.05. From eqn. 24,  $y = 9.9607$ .

## 4 Extreme Normal Deviates

```
double*
extremes(x,n)
double *x;
int n;
Returns [x_n - x_bar, x_1 - x_bar]'
```

Let  $x_1 \leq x_2 \leq \dots \leq x_n$  be the  $n$  observations. Given a known normal deviation  $\sigma$ , the largest and smallest deviation from a normal population may be computed:

$$u_n = \frac{x_n - \bar{x}}{\sigma} \quad (26)$$

and

$$u_1 = -\frac{x_1 - \bar{x}}{\sigma}, \quad (27)$$

respectively. These statistics are potentially useful for detecting outliers for populations with a known  $\sigma$  but an unknown mean. Table 25 in Pearson and Hartley [25] gives percentage points for this statistic. Pearson and Hartley [25] also give examples of the use of extreme deviates when an estimator of  $\sigma$  (independent of the sample) is known and when a combined “internal” and “external” estimate is used.

## 5 EDF Statistics for Testing Normality

[Note: This section follows closely the presentation of Stephens [35].]

Let  $x_1 \leq x_2 \leq \dots \leq x_n$  be the  $n$  observations. Suppose that the continuous distribution of  $x$  is  $F(x)$ . The empirical distribution function (EDF) is  $F_n(x)$  defined by:

$$F_n(x) = \frac{1}{n} (\text{number of observations} \leq x); \quad -\infty < x < \infty \quad (28)$$

or

$$\begin{aligned} F_n(x) &= 0, & x < x_1 \\ F_n(x) &= \frac{i}{n}, & x_i \leq x < x_{i+1}, \quad i = 1, \dots, n-1 \\ F_n(x) &= 1, & x_n \leq x. \end{aligned}$$

Thus  $F_n(x)$  is a step function calculated from the data. As  $n \rightarrow \infty$ ,  $|F_n(x) - F(x)|$  decreases to zero with probability one [35].

EDF statistics that measure the difference between  $F_n(x)$  and  $F(x)$  are divided into two classes: supremum and quadratic. On a graph of  $F_n(x)$  and  $F(x)$  versus  $x_i$ , denote the largest vertical distance when  $F_n(x) > F(x)$  as  $D^+$ . Also, let  $D^-$  denote the largest vertical distance when  $F_n(x) < F(x)$ . These two measures are supremum statistics. Quadratic statistics are given by the Cramér-von Mises family

$$Q = n \int_{-\infty}^{\infty} (F_n(x) - F(x))^2 \psi(x) dF(x) \quad (29)$$

where  $\psi(x)$  is a weighting function [35].

To compute these statistics, the Probability Integral Transformation is used:  $z = F(x)$  where  $F(x)$  is the Gaussian distribution. The new variable,  $z$ , is uniformly distributed between 0 and 1. Then  $z$  has distribution function  $F^*(z) = z$ ,  $0 \leq z \leq 1$ . A sample  $x_1, x_2, \dots, x_n$  gives values  $z_i = F(x_i)$ ,  $i = 1, \dots, n$ , and  $F_n^*(z)$  is the EDF of values  $z_i$ . For testing normality,

$$z_{(i)} = \Phi((x_{(i)} - \hat{\mu}) / \hat{\sigma}) \quad (30)$$

where  $\hat{\mu}$  and  $\hat{\sigma}$  are estimated from the data and  $\Phi(\cdot)$  denotes the cumulative probability of a standard normal. For testing if the data follows an exponential distribution  $\text{Exp}(\alpha, \beta)$ , where  $\alpha$  is known to be zero,  $\hat{\beta}$  is estimated by  $\bar{x}$  (the sample mean) and

$$z_{(i)} = 1 - \exp(-x_{(i)} / \bar{x}). \quad (31)$$

Now, EDF statistics can be computed by comparing  $F_n^*(z)$  and a uniform distribution for  $z$ . These take the same values as comparisons between  $F_n(x)$  and  $F(x)$ :

$$F_n(x) - F(x) = F_n^*(z) - F^*(z) = F_n^*(z) - z. \quad (32)$$

After ordering  $z$ -values,  $z_{(1)} \leq z_{(2)} \leq \dots \leq z_{(n)}$  and computing  $\bar{z} = \sum_{i=1}^n z_i / n$ , the supremum statistics are

$$D^+ = \max_{i=1, \dots, n} (i/n - z_{(i)}) \quad (33)$$

and

$$D^- = \max_{i=1, \dots, n} (z_{(i)} - (i-1)/n). \quad (34)$$

## 5.1 Kolmogorov $D$

```
double*  
kolmogorov_smirnov(x,n)  
double *x;  
int n;  
Returns [Dn, D]′.
```

```
double*  
kolmogorov_smirnov_exp(x,n)  
double *x;  
int n;  
Returns [De, D]′.
```

The most well-known EDF statistic is Kolmogorov's  $D$ , computed from supremum statistics:

$$D = \sup_x |F_n(x) - F(x)| = \max(D^+, D^-). \quad (35)$$

The modified form for testing a completely specified distribution [35]:

$$D^* = D(\sqrt{n} + 0.12 + 0.11/\sqrt{n}). \quad (36)$$

For testing a normal distribution with  $\mu$  and  $\sigma$  unknown, the modified equation is [35]:

$$D^n = D(\sqrt{n} - 0.01 + 0.85/\sqrt{n}). \quad (37)$$

For testing an exponential distribution with  $\alpha$  and  $\beta$  unknown,  $D$  does not need modified [35].

---

*Example:* For the sample data given in Table 4 ( $n = 584$ ),  $D^n = 4.0314$  and  $y =$ . Suppose that we wish to test the hypothesis of normality:

$H_0$ : normality  
versus the two-sided alternative  
 $H_1$ : non-normality  
at a level of significance of 0.05.

---

## 5.2 Kuiper's $V$

```
double*  
kuipers.v(x,n)  
double *x;  
int n;  
Returns [Vn, V]′.
```



```

double*
kuipers_v_exp(x,n)
double *x;
int n;
Returns [Ve, V]'.

```

Kuiper's [24]  $V$  is another statistic computed from supremum statistics:

$$V = D^+ + D^-. \quad (38)$$

The modified form for testing a completely specified distribution [35]:

$$V^* = V (\sqrt{n} + 0.155 + 0.24\sqrt{n}). \quad (39)$$

For testing a normal distribution with  $\mu$  and  $\sigma$  unknown, the modified equation is [35]:

$$V^n = V (\sqrt{n} + 0.05 + 0.82/\sqrt{n}). \quad (40)$$

For testing an exponential distribution with  $\alpha$  and  $\beta$  unknown,  $V$  the modified equation is [35]:

$$V^e = (V - 0.2/\sqrt{n}) (\sqrt{n} + 0.24 + 0.35/\sqrt{n}). \quad (41)$$

### 5.3 Pyke's Statistics

For some purposes, eqns. 33 and 34 may be modified to [26]:

$$C^+ = \max_{0 \leq i \leq n} \left( \frac{i}{n+1} - z_{(i)} \right), \quad z_{(0)} = 0, \quad (42)$$

and

$$C^- = \max_{0 \leq i \leq n} \left( z_{(i)} - \frac{i}{n+1} \right) \quad (43)$$

(following the modification of notation by Durbin [16]). Then,

$$C = \max(C^+, C^-). \quad (44)$$

Durbin [16] notes that these modifications to eqns. 33 and 34 are related to the fact that  $E(z_{(i)}) = i/(n+1)$ . Percentage points were given by Durbin [15].

### 5.4 Brunk's $B$

As an alternative to Kuiper's  $V$  (eqn. 38), Brunk [6] suggests:

$$B = C^+ + C^- \quad (45)$$

where  $C^+$  and  $C^-$  are given by eqns. 42 and 43.

## 5.5 Cramér–von Mises $W^2$

```
double*
cramer_von_mises(x,n)
double *x;
int n;
Returns [W2,n, W2]'.
```

```
double*
cramer_von_mises_exp(x,n)
double *x;
int n;
Returns [W2,e, W2]'.
```

Quadratic statistics are computed from the Cramér–von Mises family given in eqn 29. When  $\psi(x) = 1$  in eqn 29, the statistic is the Cramér–von Mises statistic  $W^2$ :

$$W^2 = \sum_{j=1}^n (Z_j - (2j-1)/(2n))^2 + \frac{1}{12n} \quad (46)$$

(When  $\psi(x) = (F(x)(1-F(x)))^{-1}$ , this yields the Anderson–Darling statistic given below in §5.7 [35].) The modified form for testing a completely specified distribution [35]:

$$W^{2,*} = (W^2 - 0.4/n + 0.6/n^2) / (1 + 1/n). \quad (47)$$

For testing a normal distribution with  $\mu$  and  $\sigma$  unknown, the modified equation is [35]:

$$W^{2,n} = W^2 (1.0 + 0.5/n). \quad (48)$$

For testing an exponential distribution with  $\alpha$  and  $\beta$  unknown, the modified equation is [35]:

$$W^{2,e} = W^2 (1.0 + 2.8/n - 3/n^2). \quad (49)$$

## 5.6 Watson $U^2$

```
double*
watson_u2(x,n)
double *x;
int n;
Returns [U2,n, U2]'.
```

```
double*
watson_u2_exp(x,n)
double *x;
int n;
Returns [U2,e, U2]'.
```

$$U^2 = W^2 - n(\bar{Z} - 0.5)^2 \quad (50)$$

where  $W^2$  is the Cramér–von Mises statistic (§5.5). The modified form for testing a completely specified distribution [35]:

$$U^{2,*} = (U^2 - 0.1/n + 0.1/n^2) / (1 + 0.8/n). \quad (51)$$

For testing a normal distribution with  $\mu$  and  $\sigma$  unknown, the modified equation is [35]:

$$U^{2,n} = U^2 (1.0 + 0.5/n). \quad (52)$$

For testing an exponential distribution with  $\alpha$  and  $\beta$  unknown, the modified equation is [35]:

$$U^{2,e} = U^2 (1.0 + 2.3/n - 3/n^2). \quad (53)$$

## 5.7 Anderson–Darling $A^2$

```
double*
anderson_darling(x,n)
double *x;
int n;
Returns [A2,n, A2]'.
```

```
double*
anderson_darling_exp(x,n)
double *x;
int n;
Returns [A2,e, A2]'.
```

Anderson and Darling [1] present another EDF test statistic which is sensitive at the tails of the distribution (rather than near the median). When  $\psi(x) = (F(x)(1 - F(x)))^{-1}$  in eqn.(29), this yields the Anderson–Darling statistic [1, 35]:

$$A^2 = -n - \frac{1}{n} \sum_{j=1}^n (2j - 1) [\ln z_j + \ln(1 - z_{n-j+1})]. \quad (54)$$

Equivalently [35],

$$A^2 = -n - \frac{1}{n} \sum_{j=1}^n [(2j - 1) \ln z_j + (2n + 1 - 2j) \ln(1 - z_j)]. \quad (55)$$

Anderson and Darling [1] give the following asymptotic significance values of  $A^2$ :

| Significance Level | Significance Point |
|--------------------|--------------------|
| 0.10               | 1.933              |
| 0.05               | 2.492              |
| 0.01               | 3.857              |

Anderson and Darling [1] state that sample size should be at least 40; however, Stephens [35] give the same asymptotic values (for more significance levels) for a sample size  $\geq 5$ .

For testing a completely specified distribution,  $A^2$  is used unmodified. For testing a normal distribution with  $\mu$  and  $\sigma$  unknown, the modified equation is [35]:

$$A^{2,n} = A^2 (1.0 + 0.75/n + 2.25/n^2). \quad (56)$$

For testing an exponential distribution with  $\alpha$  and  $\beta$  unknown, the modified equation is [35]:

$$A^{2,e} = A^2 (1.0 + 5.4/n - 11/n^2). \quad (57)$$

## 5.8 Durbin's Exact Test

```
double*
durbins_exact(x,n)
double *x;
int n;
Returns [Km, √nKm]'.
```

Durbin [14] presented a modified Kolmogorov test. The discussion that follows has been adapted from Durbin's work [14].

Let  $x_1, x_2, \dots, x_n$  be the  $n$  i.i.d. observations and suppose that it is desired to test the hypothesis that they come from the continuous distribution  $F(x)$ . If the null hypothesis is true, then  $u_j = F(x_j)$  ( $j = 1, \dots, n$ ) are independent  $U(0, 1)$  variables and are randomly scattered on the  $(0, 1)$  interval. Clustering may indicate a departure from the null hypothesis. Denoting the ordered  $u$ 's by  $0 \leq u_{(1)} \leq \dots \leq u_{(n)} \leq 1$ , let  $c_1 = u_{(1)}$ ,  $c_2 = u_{(j)} - u_{(j-1)}$  ( $j = 2, \dots, n$ ), and  $c_{n+1} = 1 - u_{(n)}$ .

Since the interest is in relative magnitudes of  $c$ 's, these are ordered:  $c_{(1)} \leq c_{(2)} \leq \dots \leq c_{(n)}$ . Then, the following transformation is applied:

$$g_j = (n + 2 - j) (c_{(j)} - c_{(j-1)}) \quad (c_{(0)} = 0; j = 1, \dots, n + 1). \quad (58)$$

Durbin [14] shows that the  $g$ 's, which depend on the *ordered* intervals, have the same distribution as the *unordered*  $c$ 's.

Letting

$$w_r = \sum_{j=1}^r g_j \quad (59)$$

it follows that  $w_1, \dots, w_n$  have the same distribution as the ordered  $U(0, 1)$  variables  $u_{(1)}, \dots, u_{(n)}$ .

From eqns. 58 and 59,  $w_j$  can be expressed as:

$$w_j = c_{(1)} + \dots + c_{(j-1)} + (n + 2 - j) c_{(j)}, \quad (j = 1, \dots, n), \quad (60)$$

where  $c_{(1)} \leq \dots \leq c_{(n)}$  is the ordered set of intervals.

In addition to two other tests, Durbin [14] introduces the *modified Kolmogorov test*. The test statistic is:

$$K_m = \max_{r=1, \dots, n} \left( \frac{r}{n} - w_r \right). \quad (61)$$

The test procedure is to reject when  $K_m$  is greater than the value tabulated for a one-sided Kolmogorov test.

---

*Example:* For the sample data given in Table 4 ( $n = 584$ ),  $K_m = 0.4127$ . To test the hypothesis of normality:

$H_0$ : normality

versus the one-sided alternative

$H_1$ : non-normality

at a level of significance of 0.05, we would reject  $H_0$  if  $K_m$  is larger than 0.895 (critical value of  $D$  for  $\alpha = 0.05$ ). Therefore, we cannot reject  $H_0$ .

---

## 6 Chi-Square Test

```
double*
chi_square(x,n)
double *x;
int n;
Returns [x2, k - 3]'
```

```
double*
chi_square_exp(x,n)
double *x;
int n;
Returns [x2, k - 2]'
```

According to Shapiro [34], the chi-square goodness of fit test is the oldest procedure for testing distributional assumptions. It is useful for testing normality and exponentiality when the number of observations is large (because its power is poor for small samples when compared to other tests). It is also useful when data are discrete [34].

The basic idea is to divide the  $n$  data into  $k$  cells and compare the observed number in each cell with the expected number in each cell. The resulting statistic is distributed as a chi-square random variable with  $k - 1 - t$  degrees of freedom, where  $t$  is the number of parameters estimated. The number of cells is taken as

$$k = (\text{int})4 \left[ 0.75 (n - 1)^2 \right]^{1/5}. \quad (62)$$

The ratio  $n/k$  should be at least 5; otherwise another test should be used [34]. In this implementation,  $k$  is decremented by one until  $n/k \geq 5$ .

Let  $x_{(1)}, x_{(2)}, \dots, x_{(k)}$  be the upper boundaries of cells. Choose  $x_{(i)}$  so that the probability of being in any cell is the same:

$$P(x \leq x_{(i)}) = \frac{i}{k}, \quad i = 1, 2, \dots, k \quad (63)$$

In this implementation, only the case of raw data, as opposed to pre-tabulated data, is considered (i.e., equal probability cells).

For testing the normality hypothesis, let  $x_{(0)} = -\infty$  and  $x_{(k)} = \infty$ . The values of  $x_{(i)}$  are:

$$x_{(i)} = \bar{x} + s Z_{i/k} \quad (64)$$

what should the notation be for rounding?  
For ceil, we use  $\lceil x \rceil$ .  
For floor, we use  $\lfloor x \rfloor$ .

where  $\bar{x}$  and  $s$  are estimated mean and variance parameters and  $Z_{i/k}$  are percentiles of the standard normal distribution. The test statistic is

$$x^2 = \frac{k}{n} \sum_{i=1}^k f_i^2 - n \quad (65)$$

where  $f_i$  is the number of observations in cell  $i$ . The hypothesis of normality is rejected at an  $\alpha$  level if  $x^2$  is greater  $x_{\alpha}^2$ , a  $\chi^2$  random variable with  $k - 3$  degrees of freedom.

---

*Example:* For the sample data given in Table 4 ( $n = 584$ ),  $x^2 = 952.7$  with  $\nu = 45$  degrees of freedom. Since  $\chi_{45,0.05}^2 \approx 30.33$  (Table 2), we reject  $H_0$  at an  $\alpha = 0.05$  level.

---

For testing the exponentiality hypothesis, let  $x_{(0)} = 0$  and  $x_{(k)} = \infty$ . The values of  $x_{(i)}$  are:

$$x_{(i)} = -\frac{1}{\lambda} \ln \left( 1 - \frac{i}{k} \right), i = 1, 2, \dots, k - 1. \quad (66)$$

The parameter  $\lambda$  is estimated from

$$\hat{\lambda} = n \left( \sum_{i=1}^n x_i \right)^{-1} \quad (67)$$

where  $x_i$  is the  $i$ th observation in the sample. Equation (65) is the statistic used for testing exponentiality. The hypothesis of exponentiality is rejected at an  $\alpha$  level if  $x^2$ , a  $\chi^2$  random variable with  $k - 2$  degrees of freedom.

---

*Example:* For the sample data given in Table 4 ( $n = 584$ ),  $x^2 = 308.11$  with  $\nu = 46$  degrees of freedom. Since  $\chi_{46,0.05}^2 \approx 31.16$  (Table 2), we reject  $H_0$  : exponentiality, at an  $\alpha = 0.05$  level.

---

## 7 Analysis of Variance Tests

### 7.1 Shapiro-Wilk $W$

```
double*
shapiro_wilk(x,n)
double *x;
int n;
Returns [W, S^2]'
```

```

double*
shapiro_wilk_exp(x,n)
double *x;
int n;
Returns [W, S2]' .

```

Recall the description of a probability plot given on page 2. Ordered observations are plotted against expected values of order statistics from the distribution being tested. The plot tends to be linear if the distributional assumption is correct. If a generalized least squares is performed, an  $F$ -type ratio could be used to test the fit of a linear model. This was the basis of test introduced by Shapiro and Wilk [32]. Foregoing many of the details in the derivation, the test procedures for normality and exponentiality are given below.

Let  $x_1 \leq x_2 \leq \dots \leq x_n$  be the  $n$  ordered observations and let

$$S^2 = \sum_{i=1}^n x_i^2 - \frac{1}{n} \left( \sum_{i=1}^n x_i \right)^2. \quad (68)$$

Calculate

$$b = \sum_{i=1}^k a_{n-i+1} (x_{n-i+1} - x_i) \quad (69)$$

where  $k = n/2$  if  $n$  is even,  $k = (n-1)/2$  if  $n$  is odd, and  $a_{n-i+1}$  are found in Table 6. Then a test of normality for small samples ( $3 \leq n \leq 50$ ) is defined as

$$W = \frac{b^2}{S^2} \quad (70)$$

Small values of  $W$  indicate non-normality (“lower-tail”). Hence if the computed value of  $W$  is less than the  $W_\alpha$  shown in Table 7, the hypothesis of normality is rejected.

---

*Example:* Using the first 40 observations from the sample data given in Table 4,  $W = 0.0000245$ . Using  $\alpha = 0.05$  and Table 7,  $W_{0.05} = 0.940$ . Since  $W < W_{0.05}$ , we reject  $H_0$ .

---

For testing exponentiality, no tabulated constants are needed for calculation of  $b$ :

$$b = \sqrt{\frac{n}{n-1}} (\bar{x} - x_1) \quad (71)$$

where

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i. \quad (72)$$

This assumes that the origin parameter is unknown. It also differs from the test of normality in that it is a two-tailed procedure. That is, too small or too large a value of the test statistic indicates non-exponentiality [34].

---

*Example:* Using the first 40 observations from the sample data given in Table 4,  $W = 0.0909$ . Using  $\alpha = 0.05$  and Table 8,  $W_{0.025} = 0.0148$  and  $W_{0.975} = 0.0447$ . Since  $W$  is not contained in the interval  $[W_{0.025}, W_{0.975}]$ , we reject  $H_0$ : exponentiality.

---

## 7.2 Modified Shapiro–Francia $W'$

```
double*
shapiro_francia(x,n)
double *x;
int n;
Returns  $[W', S^2]'$ .
```

The  $W$  test of normality in the previous section for sample sizes of 50 or less. Shapiro and Francia [31] extended the  $W$  test for  $n$  up to 99 by replacing the values  $a_{n-i+1}$  in Table 6  $b_{n-i+1}$  in Table 9. The test procedure follows.

Let  $x_1 \leq x_2 \leq \dots \leq x_n$  be the  $n$  ordered observations. Then a test of normality for large samples is defined as:

$$W' = \frac{b'}{S^2} \quad (73)$$

The numerator  $b'$  is defined as:

$$b' = \sum_{i=1}^k b_{n-i+1} (x_{n-i+1} - x_i) \quad (74)$$

where  $k = n/2$  if  $n$  is even and  $k = (n-1)/2$  if  $n$  is odd. Significant values, determined empirically by Shapiro and Francia [31] are given in Table 10. D'Agostino [10] notes that the values given by Shapiro and Francia [31] in the lower tail were “higher than what they should be” since too few samples were used in determining these significance levels.

---

*Example:* Using the first 99 observations from the sample data given in Table 4,  $W' = 1.0139$ . Using  $\alpha = 0.05$  and Table 10,  $W'_{0.05} = 0.976$ . Since  $W' > W'_{0.05}$ , we cannot reject  $H_0$ .

---

## 7.3 Weisberg-Bingham $\tilde{W}'$

```
double*
weisberg_bingham(x,n)
double *x;
int n;
Returns  $[\tilde{W}', S^2]'$ .
```



An alternative way of computing  $b'$  is to note that the vector  $[b_1, b_2, \dots, b_n]'$  is equivalent to  $m' / (m'm)^{1/2}$  where  $m' = (m_1, m_2, \dots, m_n)$  denotes a vector of expected normal order statistics. One approximation for normal order statistics attributed to Blom [3] is:

$$E(r, n) = -\Phi^{-1} \left( \frac{r - \alpha}{n - 2\alpha + 1} \right) \quad (75)$$

with a recommended “compromise value  $\alpha = 0.375$  [28].” Define this new statistic as  $\tilde{W}'$ . So, instead of hardcoding constants (as done in §7.1-7.2), this approximation is used. Since  $\tilde{W}'$  is essentially the same as  $W'$ , the table of critical values for  $W'$  (Table 10) may be used.

#### 7.4 D’Agostino’s $D$ Test of Normality

```
double*
dagostino.d(x,n)
double *x;
int n;
Returns [D, y]'.
```

D’Agostino [10] presents a modified Shapiro-Wilk  $W$  test that eliminates the need for a table of weights. The test statistic is given as

$$\begin{aligned} D &= T / (n^2 \sqrt{m_2}) \\ &= T / \left( n^{3/2} \sqrt{\sum_{j=1}^n (x_j - \bar{x})^2} \right) \end{aligned} \quad (76)$$

where

$$T = \sum_{i=1}^n \left( i - \frac{1}{2} (n + 1) \right) x_i. \quad (77)$$

An approximate standard variable is

$$y = \frac{\sqrt{n} (D - 0.28209479)}{0.02998598}. \quad (78)$$

Significant values are given in Table 3.

---

*Example:* For the sample data given in Table 4 ( $n = 584$ ),  $D = 0.2859$  and  $y = 3.0667$ . Suppose that we wish to test the hypothesis of normality:

$H_0$ : normality

versus the two-sided alternative

$H_1$ : non-normality

at a level of significance of 0.005. From Table 3 (linearly interpolating), we reject  $H_0$  if  $y < -3.006$  or  $y > 2.148$ . Therefore, we cannot reject  $H_0$ .

---

## 7.5 Royston's Modification

```
double*
royston(x,n)
double *x;
int n;
Returns [W, P]'
```

Royston [30] also presented a modified  $W$  statistic for  $n$  up to 2000 that did not require extensive use of tabulated constants. If  $m' = (m_1, m_2, \dots, m_n)$  denotes a vector of expected values of standard normal order statistics and  $V = (v_{ij})$  denote the corresponding  $n \times n$  covariance matrix, then  $W$  may be written as:

$$W = \left[ \sum_{i=1}^n a_i x_{(i)} \right]^2 / \sum_{i=1}^n (x_{(i)} - \bar{x})^2 \quad (79)$$

where

$$a' = m'V^{-1} [(m'V^{-1})(V^{-1}m')]^{1/2}. \quad (80)$$

Let  $a^* = m'V^{-1}$ ; The following approximation for  $a^*$  is used:

$$\hat{a}^* = \begin{cases} 2m_i, & i=2,3,\dots,n-1 \\ \left( \frac{\hat{a}_1^2}{1-2\hat{a}_1^2} \sum_{i=2}^{n-1} \hat{a}_i^{*2} \right)^{1/2}, & i=1, i=n \end{cases} \quad (81)$$

where

$$\hat{a}_1^2 = \hat{a}_n^2 = \begin{cases} g(n-1), & n \leq 20 \\ g(n), & n > 20 \end{cases} \quad (82)$$

and

$$g(n) = \frac{\Gamma(\frac{1}{2}[n+1])}{\sqrt{2}\Gamma(\frac{1}{2}n+1)}. \quad (83)$$

The function  $g(n)$  is approximated using:

$$g(n) = \left[ \frac{6n+7}{6n+13} \right] \left( \frac{\exp(1)}{n+2} \left[ \frac{n+1}{n_2} \right]^{n-2} \right)^{1/2} \quad (84)$$

Royston [30] used eqns. 81–84 for the range  $7 \leq n \leq 2000$ , but exact values of  $a_i$  for  $n < 7$ .

Royston [30] used the following normalizing transformation:

$$y = (1 - W)^\lambda \quad (85)$$

so that

$$z = \left[ (1 - W)^\lambda - \mu_y \right] / \sigma_y \quad (86)$$

can be compared with the upper tail of a standard normal. Large values of  $z$  indicate non-normality of the original sample.

This implementation in `cdhc` closely follows Royston's published FORTRAN code [29, 28]. It returns  $W$  and a corresponding  $P$  value (smallest level at which we could have preset  $\alpha$  and still have been able to reject  $H_0$ ). It also utilizes algorithms by Hill [19] and Wichura [36].

## 8 Kotz Separate Families $T'_f$

```

double*
kotz_families(x,n)
double *x;
int n;
Returns [T'_f, T_f]'.

```

Kotz [23] developed a test where the null hypothesis  $H_0$  is that the sample  $x_1, x_2, \dots, x_n$  came from a lognormal distribution, and the alternate hypothesis is that the parent population was normal. The test statistic, given as:

$$T'_f = \frac{\log \frac{\hat{\beta}_2}{\beta_{2,\hat{\alpha}}}}{2\sqrt{n}\left\{\frac{1}{4}(e^{4\hat{\alpha}_2} + 2e^{3\hat{\alpha}_2} - 4) - \hat{\alpha}_2 - \frac{\hat{\alpha}_2(2e^{\hat{\alpha}_2} - 1)^2}{2(2e^{\hat{\alpha}_2} - 1)^2} + \frac{3}{4}e^{\hat{\alpha}_2}\right\}^{1/2}} \quad (87)$$

is asymptotically normal [8].

---

*Example:* For the sample data given in Table 4 ( $n = 584$ ),  $T'_f = -0.6021$ . Suppose that we wish to test the hypothesis

$H_0$  : lognormal

versus

$H_1$  : normal

at a level of significance of 0.05. We would reject  $H_0$  if  $T'_f$  is larger than 1.645. Therefore, we reject  $H_0$ .

---

The discussion that follows explains in more detail how this statistic is calculated and how it was derived. The remainder of this section was taken directly from the work of Kotz [23] (pages 123,124–126).

... A test for this special situation was considered by Roy [27], where he bases his decision on the statistic

$$R = \frac{L_l}{L_n} \quad (88)$$

where  $L_l$  denotes the likelihood of the sample under the lognormal hypothesis and  $L_n$  that under the normal hypothesis. If  $R > 1$  one accepts lognormality, and if  $R < 1$  normality is accepted. More recently Cox [7, 8] has elaborated on Roy's heuristic approach, and has derived a general class of tests to discriminate between hypotheses that are *separate* (in the sense that an arbitrary simple hypothesis in  $H_0$  cannot be obtained as a limit—in the parameter space—of a simple hypothesis in  $H_1$ ). We will now apply Cox's general theory to testing lognormality against normality...

Suppose  $x_1, x_2, \dots, x_n$  is a random sample from a certain population. The null hypothesis,  $H_f$ , is that the p.d.f. of the  $x$ 's is log-normal and the alternate hypothesis,  $H_g$ , is that the p.d.f. is normal, that is, for  $H_f$

$$f(y, \beta) = \frac{1}{\sqrt{2\pi\beta}} \exp - \left( \frac{(\log y - \beta)^2}{2\beta} \right), \quad -\infty < y < \infty. \quad (89)$$

and for  $H_g$ :

$$g(y, \alpha) = \frac{1}{y\sqrt{2\pi\alpha_2}} \exp - \left( \frac{(y - \alpha_1)^2}{2\alpha_2} \right), \quad y > 0. \quad (90)$$

From the maximum likelihood equations we find that

$$\hat{\alpha}_1 = \frac{1}{n} \sum \log x_i; \quad \hat{\alpha}_2 = \frac{1}{n} \sum (\log x_i - \hat{\alpha}_1), \quad (91)$$

and analogous equations for  $\hat{\beta}_1$  and  $\hat{\beta}_2$ .

Under  $H_f$ , the log-normal null hypothesis, as the sample size  $n$  increases to infinity,  $\hat{\alpha}_1 \rightarrow \alpha_1$ ,  $\hat{\alpha}_2 \rightarrow \alpha_2$ ,  $\hat{\beta}_{1,\alpha} \rightarrow \beta_{1,\alpha}$ , and  $\hat{\beta}_{2,\alpha} \rightarrow \beta_{2,\alpha}$  where

$$\hat{\beta}_{1,\alpha} = \exp \left( \alpha_1 + \frac{\alpha_2}{2} \right) \quad (92)$$

and

$$\hat{\beta}_{2,\alpha} = \exp(2\alpha_1 + \alpha_2) [\exp(\alpha_2) - 1]. \quad (93)$$

Cox's test is based on the log likelihood ratio

$$L_{fg} = \sum_{i=1}^n \log \frac{f(x_i, \hat{\alpha})}{g(x_i, \hat{\beta})} \quad (94)$$

and his test statistic is given by

$$T_f = L_{fg} - E_{\hat{\alpha}}(L_{fg}) \quad (95)$$

where  $E_{\hat{\alpha}}(L_{fg})$  is the expected value under  $H_f$  when  $\alpha$  takes the value  $\hat{\alpha}$ . Writing

$$F = \log f(x, \alpha), \quad F_{\alpha_i} = \frac{\partial \log f(x, \alpha)}{\partial \alpha_i}, \quad i = 1, 2 \quad (96)$$

$$F_{\alpha_i \alpha_j} = \frac{\partial^2 \log f(x, \alpha)}{\partial \alpha_i \partial \alpha_j}, \quad G = \log g(x, \beta) \quad (97)$$

$$G_{\beta_i} = \frac{\partial \log g(x, \beta)}{\partial \beta_i}, \quad \text{etc.}, \quad (98)$$

Cox shows that  $T_f$  is asymptotically normal with zero mean and variance

$$V_{\alpha}(T_f) = nV_{\alpha}(F - G) - \sum \frac{C_{\alpha}^2(F - G, F_{\alpha_i})}{V_{\alpha}(F_{\alpha_i})} \quad (99)$$

where  $V_{\alpha}(\cdot)$ ,  $C_{\alpha}(\cdot)$ , denote variance and covariance under  $H_f$ .

In our case it can be shown that

$$T_f = \frac{n}{2} \log \frac{\hat{\beta}_2}{\hat{\beta}_{2,\hat{\alpha}}} \quad (100)$$

Results of the following type are used in the derivation of  $V_{\alpha}(T_f)$ :

$$E_{\alpha}[x^2 \log x] = (\alpha_1 + 2\alpha_2) \exp(2\alpha_1 + 2\alpha_2) \quad (101)$$

$$E_\alpha [x^2 \log^2 x] = (\alpha_2 + \alpha_1^2 + 4\alpha_1\alpha_2 + 4\alpha_2^2) \exp(2\alpha_1 + 2\alpha_2) \quad (102)$$

$$E_\alpha [(\log x)(\log x - \alpha)] = \alpha_2 \quad (103)$$

$$E_\alpha [(\log x)(\log x - \beta_1)^2] = \beta_2(\alpha_1 + 2\alpha_2) \quad (104)$$

$$E_\alpha [(\log x - \alpha_1)(\log x - \beta_1)^2] = 2\alpha_2\beta_2. \quad (105)$$

Using these results, after a considerable amount of simplification, we get

$$V_\alpha(T_f) = n \left[ \frac{1}{4} (e^{4\alpha_2} + 2e^{3\alpha_2} + 3e^{\alpha_2} - 4) \alpha_2 - \frac{\alpha_2 (2e^{\alpha_2} - 1)^2}{2(2e^{\alpha_2} - 1)^2} \right] \quad (106)$$

Cox [8] has shown that

$$T'_f = \frac{T_f}{\sqrt{V_\alpha(T_f)}} \quad (107)$$

is asymptotically standardized normal. In our case we get, after substituting the estimators for the parameters,

$$T'_f = \frac{\log \frac{\hat{\beta}_2}{\hat{\beta}_{2,\hat{\alpha}}}}{2\sqrt{n} \left\{ \frac{1}{4} (e^{4\hat{\alpha}_2} + 2e^{3\hat{\alpha}_2} - 4) - \hat{\alpha}_2 - \frac{\hat{\alpha}_2 (2e^{\hat{\alpha}_2} - 1)^2}{2(2e^{\hat{\alpha}_2} - 1)^2} + \frac{3}{4} e^{\hat{\alpha}_2} \right\}^{1/2}} \quad (108)$$

## 9 Utility Functions

This section describes some useful functions included in `cdhc` but not necessarily described in the previous sections, e.g., normal order statistics, normal probabilities, inverse normals.

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Table 1: Cumulative Standard Normal Distribution.  
Area Under the Normal Curve from

$$-\infty \text{ to } z = \frac{X_i - \mu}{\sigma}.$$

Computed by the author using algorithm 5666 for the error function, from Hart *et al.* [18].

| $z$ | 0.00    | 0.01    | 0.02    | 0.03    | 0.04    | 0.05    | 0.06    | 0.07    | 0.08    | 0.09    |
|-----|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 0.0 | 0.50000 | 0.50399 | 0.50798 | 0.51197 | 0.51595 | 0.51994 | 0.52392 | 0.52790 | 0.53188 | 0.53586 |
| 0.1 | 0.53983 | 0.54380 | 0.54776 | 0.55172 | 0.55567 | 0.55962 | 0.56356 | 0.56749 | 0.57142 | 0.57535 |
| 0.2 | 0.57926 | 0.58317 | 0.58706 | 0.59095 | 0.59483 | 0.59871 | 0.60257 | 0.60642 | 0.61026 | 0.61409 |
| 0.3 | 0.61791 | 0.62172 | 0.62552 | 0.62930 | 0.63307 | 0.63683 | 0.64058 | 0.64431 | 0.64803 | 0.65173 |
| 0.4 | 0.65542 | 0.65910 | 0.66276 | 0.66640 | 0.67003 | 0.67364 | 0.67724 | 0.68082 | 0.68439 | 0.68793 |
| 0.5 | 0.69146 | 0.69497 | 0.69847 | 0.70194 | 0.70540 | 0.70884 | 0.71226 | 0.71566 | 0.71904 | 0.72240 |
| 0.6 | 0.72575 | 0.72907 | 0.73237 | 0.73565 | 0.73891 | 0.74215 | 0.74537 | 0.74857 | 0.75175 | 0.75490 |
| 0.7 | 0.75804 | 0.76115 | 0.76424 | 0.76730 | 0.77035 | 0.77337 | 0.77637 | 0.77935 | 0.78230 | 0.78524 |
| 0.8 | 0.78814 | 0.79103 | 0.79389 | 0.79673 | 0.79955 | 0.80234 | 0.80511 | 0.80785 | 0.81057 | 0.81327 |
| 0.9 | 0.81594 | 0.81859 | 0.82121 | 0.82381 | 0.82639 | 0.82894 | 0.83147 | 0.83398 | 0.83646 | 0.83891 |
| 1.0 | 0.84134 | 0.84375 | 0.84614 | 0.84849 | 0.85083 | 0.85314 | 0.85543 | 0.85769 | 0.85993 | 0.86214 |
| 1.1 | 0.86433 | 0.86650 | 0.86864 | 0.87076 | 0.87286 | 0.87493 | 0.87698 | 0.87900 | 0.88100 | 0.88298 |
| 1.2 | 0.88493 | 0.88686 | 0.88877 | 0.89065 | 0.89251 | 0.89435 | 0.89617 | 0.89796 | 0.89973 | 0.90147 |
| 1.3 | 0.90320 | 0.90490 | 0.90658 | 0.90824 | 0.90988 | 0.91149 | 0.91309 | 0.91466 | 0.91621 | 0.91774 |
| 1.4 | 0.91924 | 0.92073 | 0.92220 | 0.92364 | 0.92507 | 0.92647 | 0.92785 | 0.92922 | 0.93056 | 0.93189 |
| 1.5 | 0.93319 | 0.93448 | 0.93574 | 0.93699 | 0.93822 | 0.93943 | 0.94062 | 0.94179 | 0.94295 | 0.94408 |
| 1.6 | 0.94520 | 0.94630 | 0.94738 | 0.94845 | 0.94950 | 0.95053 | 0.95154 | 0.95254 | 0.95352 | 0.95449 |
| 1.7 | 0.95543 | 0.95637 | 0.95728 | 0.95818 | 0.95907 | 0.95994 | 0.96080 | 0.96164 | 0.96246 | 0.96327 |
| 1.8 | 0.96407 | 0.96485 | 0.96562 | 0.96638 | 0.96712 | 0.96784 | 0.96856 | 0.96926 | 0.96995 | 0.97062 |
| 1.9 | 0.97128 | 0.97193 | 0.97257 | 0.97320 | 0.97381 | 0.97441 | 0.97500 | 0.97558 | 0.97615 | 0.97670 |
| 2.0 | 0.97725 | 0.97778 | 0.97831 | 0.97882 | 0.97932 | 0.97982 | 0.98030 | 0.98077 | 0.98124 | 0.98169 |
| 2.1 | 0.98214 | 0.98257 | 0.98300 | 0.98341 | 0.98382 | 0.98422 | 0.98461 | 0.98500 | 0.98537 | 0.98574 |
| 2.2 | 0.98610 | 0.98645 | 0.98679 | 0.98713 | 0.98745 | 0.98778 | 0.98809 | 0.98840 | 0.98870 | 0.98899 |
| 2.3 | 0.98928 | 0.98956 | 0.98983 | 0.99010 | 0.99036 | 0.99061 | 0.99086 | 0.99111 | 0.99134 | 0.99158 |
| 2.4 | 0.99180 | 0.99202 | 0.99224 | 0.99245 | 0.99266 | 0.99286 | 0.99305 | 0.99324 | 0.99343 | 0.99361 |
| 2.5 | 0.99379 | 0.99396 | 0.99413 | 0.99430 | 0.99446 | 0.99461 | 0.99477 | 0.99492 | 0.99506 | 0.99520 |
| 2.6 | 0.99534 | 0.99547 | 0.99560 | 0.99573 | 0.99585 | 0.99598 | 0.99609 | 0.99621 | 0.99632 | 0.99643 |
| 2.7 | 0.99653 | 0.99664 | 0.99674 | 0.99683 | 0.99693 | 0.99702 | 0.99711 | 0.99720 | 0.99728 | 0.99736 |
| 2.8 | 0.99744 | 0.99752 | 0.99760 | 0.99767 | 0.99774 | 0.99781 | 0.99788 | 0.99795 | 0.99801 | 0.99807 |
| 2.9 | 0.99813 | 0.99819 | 0.99825 | 0.99831 | 0.99836 | 0.99841 | 0.99846 | 0.99851 | 0.99856 | 0.99861 |
| 3.0 | 0.99865 | 0.99869 | 0.99874 | 0.99878 | 0.99882 | 0.99886 | 0.99889 | 0.99893 | 0.99896 | 0.99900 |
| 3.1 | 0.99903 | 0.99906 | 0.99910 | 0.99913 | 0.99916 | 0.99918 | 0.99921 | 0.99924 | 0.99926 | 0.99929 |
| 3.2 | 0.99931 | 0.99934 | 0.99936 | 0.99938 | 0.99940 | 0.99942 | 0.99944 | 0.99946 | 0.99948 | 0.99950 |
| 3.3 | 0.99952 | 0.99953 | 0.99955 | 0.99957 | 0.99958 | 0.99960 | 0.99961 | 0.99962 | 0.99964 | 0.99965 |
| 3.4 | 0.99966 | 0.99968 | 0.99969 | 0.99970 | 0.99971 | 0.99972 | 0.99973 | 0.99974 | 0.99975 | 0.99976 |
| 3.5 | 0.99977 | 0.99978 | 0.99978 | 0.99979 | 0.99980 | 0.99981 | 0.99981 | 0.99982 | 0.99983 | 0.99983 |



Table 2: Cumulative Chi-Square Distribution.

Computed by the author using CDFLIB [5], with the exception of items marked with a dagger (†), which were found in *Biometrika Tables for Statisticians* (1966), 3rd. Ed., University College, London, as cited by Shapiro [34].

| $\nu$ | $\alpha$   |           |           |         |        |       |       |       |       |       |
|-------|------------|-----------|-----------|---------|--------|-------|-------|-------|-------|-------|
|       | 0.995      | 0.990     | 0.975     | 0.950   | 0.900  | 0.100 | 0.050 | 0.025 | 0.010 | 0.005 |
| 1     | 0.0000393† | 0.000157† | 0.000982† | 0.0158† | 0.102† | 2.71  | 3.84  | 5.02  | 6.63  | 7.88  |
| 2     | 0.0100     | 0.0201    | 0.0506    | 0.103   | 0.211  | 4.61  | 5.99  | 7.38  | 9.21  | 10.6  |
| 3     | 0.0717     | 0.115     | 0.216     | 0.352   | 0.584  | 6.25  | 7.81  | 9.35  | 11.3  | 12.8  |
| 4     | 0.207      | 0.297     | 0.484     | 0.711   | 1.06   | 7.78  | 9.49  | 11.1  | 13.3  | 14.9  |
| 5     | 0.412      | 0.554     | 0.831     | 1.15    | 1.61   | 9.24  | 11.1  | 12.8  | 15.1  | 16.8  |
| 6     | 0.676      | 0.872     | 1.24      | 1.64    | 2.20   | 10.6  | 12.6  | 14.5  | 16.8  | 18.5  |
| 7     | 0.989      | 1.24      | 1.69      | 2.17    | 2.83   | 12.0  | 14.1  | 16.0  | 18.5  | 20.3  |
| 8     | 1.34       | 1.65      | 2.18      | 2.73    | 3.49   | 13.4  | 15.5  | 17.5  | 20.1  | 22.0  |
| 9     | 1.73       | 2.09      | 2.70      | 3.33    | 4.17   | 14.7  | 16.9  | 19.0  | 21.7  | 23.6  |
| 10    | 2.16       | 2.56      | 3.25      | 3.94    | 4.87   | 16.0  | 18.3  | 20.5  | 23.2  | 25.2  |
| 11    | 2.60       | 3.05      | 3.82      | 4.57    | 5.58   | 17.3  | 19.7  | 21.9  | 24.7  | 26.8  |
| 12    | 3.07       | 3.57      | 4.40      | 5.23    | 6.30   | 18.6  | 21.0  | 23.3  | 26.2  | 28.3  |
| 13    | 3.57       | 4.11      | 5.01      | 5.89    | 7.04   | 19.8  | 22.4  | 24.7  | 27.7  | 29.8  |
| 14    | 4.07       | 4.66      | 5.63      | 6.57    | 7.79   | 21.1  | 23.7  | 26.1  | 29.1  | 31.3  |
| 15    | 4.60       | 5.23      | 6.26      | 7.26    | 8.55   | 22.3  | 25.0  | 27.5  | 30.6  | 32.8  |
| 16    | 5.14       | 5.81      | 6.91      | 7.96    | 9.31   | 23.5  | 26.3  | 28.9  | 32.0  | 34.3  |
| 17    | 5.70       | 6.41      | 7.56      | 8.67    | 10.1   | 24.8  | 27.6  | 30.2  | 33.4  | 35.7  |
| 18    | 6.26       | 7.01      | 8.23      | 9.39    | 10.9   | 26.0  | 28.9  | 31.5  | 34.8  | 37.2  |
| 19    | 6.84       | 7.63      | 8.91      | 10.1    | 11.7   | 27.2  | 30.1  | 32.9  | 36.2  | 38.6  |
| 20    | 7.43       | 8.26      | 9.59      | 10.9    | 12.4   | 28.4  | 31.4  | 34.2  | 37.6  | 40.0  |
| 21    | 8.03       | 8.90      | 10.3      | 11.6    | 13.2   | 29.6  | 32.7  | 35.5  | 38.9  | 41.4  |
| 22    | 8.64       | 9.54      | 11.0      | 12.3    | 14.0   | 30.8  | 33.9  | 36.8  | 40.3  | 42.8  |
| 23    | 9.26       | 10.2      | 11.7      | 13.1    | 14.9   | 32.0  | 35.1  | 38.0  | 41.6  | 44.2  |
| 24    | 9.89       | 10.9      | 12.4      | 13.9    | 15.7   | 33.2  | 36.4  | 39.4  | 43.0  | 45.6  |
| 25    | 10.5       | 11.5      | 13.1      | 14.6    | 16.5   | 34.4  | 37.7  | 40.6  | 44.3  | 46.9  |
| 26    | 11.2       | 12.2      | 13.8      | 15.4    | 17.3   | 35.6  | 38.9  | 41.9  | 45.6  | 48.3  |
| 27    | 11.8       | 12.9      | 14.6      | 16.2    | 18.1   | 36.7  | 40.1  | 43.2  | 47.0  | 49.6  |
| 28    | 12.5       | 13.6      | 15.3      | 16.9    | 18.9   | 37.9  | 41.3  | 44.5  | 48.3  | 51.0  |
| 29    | 13.1       | 14.3      | 16.0      | 17.7    | 19.8   | 39.1  | 42.6  | 45.7  | 49.6  | 52.3  |
| 30    | 13.8       | 15.0      | 16.8      | 18.5    | 20.6   | 40.3  | 43.8  | 47.0  | 50.9  | 53.7  |

According to Shapiro [34], for situations with larger than 30 degrees of freedom,  $\chi^2_{\nu,\alpha} = 0.5(z_\alpha + \sqrt{2\nu - 1})^2$ , where  $z_\alpha$  is the 100 $\alpha$ % point of the standard normal distribution, e.g.,  $z_{0.05} = -1.645$  from Table 1.

Table 3: Significant Values of D'Agostino's D Test ( $y$  statistic of eqn. 78).  
 Reproduced from D'Agostino [10].

| n    | Percentiles |        |        |        |        |       |       |       |       |       |
|------|-------------|--------|--------|--------|--------|-------|-------|-------|-------|-------|
|      | 0.5         | 1.0    | 2.5    | 5      | 10     | 90    | 95    | 97.5  | 99    | 99.5  |
| 10   | -4.66       | -4.06  | -3.25  | -2.62  | -1.99  | 0.149 | 0.235 | 0.299 | 0.356 | 0.385 |
| 12   | -4.63       | -4.02  | -3.20  | -2.58  | -1.94  | 0.237 | 0.329 | 0.381 | 0.440 | 0.479 |
| 14   | -4.57       | -3.97  | -3.16  | -2.53  | -1.90  | 0.308 | 0.399 | 0.460 | 0.515 | 0.555 |
| 16   | -4.52       | -3.92  | -3.12  | -2.50  | -1.87  | 0.367 | 0.459 | 0.526 | 0.587 | 0.613 |
| 18   | -4.47       | -3.87  | -3.08  | -2.47  | -1.85  | 0.417 | 0.515 | 0.574 | 0.636 | 0.667 |
| 20   | -4.41       | -3.83  | -3.04  | -2.44  | -1.82  | 0.460 | 0.565 | 0.628 | 0.690 | 0.720 |
| 22   | -4.36       | -3.78  | -3.01  | -2.41  | -1.81  | 0.497 | 0.609 | 0.677 | 0.744 | 0.775 |
| 24   | -4.32       | -3.75  | -2.98  | -2.39  | -1.79  | 0.530 | 0.648 | 0.720 | 0.783 | 0.822 |
| 26   | -4.27       | -3.71  | -2.96  | -2.37  | -1.77  | 0.559 | 0.682 | 0.760 | 0.827 | 0.867 |
| 28   | -4.23       | -3.68  | -2.93  | -2.35  | -1.76  | 0.586 | 0.714 | 0.797 | 0.868 | 0.910 |
| 30   | -4.19       | -3.64  | -2.91  | -2.33  | -1.75  | 0.610 | 0.743 | 0.830 | 0.906 | 0.941 |
| 32   | -4.16       | -3.61  | -2.88  | -2.32  | -1.73  | 0.631 | 0.770 | 0.862 | 0.942 | 0.983 |
| 34   | -4.12       | -3.59  | -2.86  | -2.30  | -1.72  | 0.651 | 0.794 | 0.891 | 0.975 | 1.02  |
| 36   | -4.09       | -3.56  | -2.85  | -2.29  | -1.71  | 0.669 | 0.816 | 0.917 | 1.00  | 1.05  |
| 38   | -4.06       | -3.54  | -2.83  | -2.28  | -1.70  | 0.686 | 0.837 | 0.941 | 1.03  | 1.08  |
| 40   | -4.03       | -3.51  | -2.81  | -2.26  | -1.70  | 0.702 | 0.857 | 0.964 | 1.06  | 1.11  |
| 42   | -4.00       | -3.49  | -2.80  | -2.25  | -1.69  | 0.716 | 0.875 | 0.986 | 1.09  | 1.14  |
| 44   | -3.98       | -3.47  | -2.78  | -2.24  | -1.68  | 0.730 | 0.892 | 1.01  | 1.11  | 1.17  |
| 46   | -3.95       | -3.45  | -2.77  | -2.23  | -1.67  | 0.742 | 0.908 | 1.02  | 1.13  | 1.19  |
| 48   | -3.93       | -3.43  | -2.75  | -2.22  | -1.67  | 0.754 | 0.923 | 1.04  | 1.15  | 1.22  |
| 50   | -3.91       | -3.41  | -2.74  | -2.21  | -1.66  | 0.765 | 0.937 | 1.06  | 1.18  | 1.24  |
| 60   | -3.81       | -3.34  | -2.68  | -2.17  | -1.64  | 0.812 | 0.997 | 1.13  | 1.26  | 1.34  |
| 70   | -3.73       | -3.27  | -2.64  | -2.14  | -1.61  | 0.849 | 1.05  | 1.19  | 1.33  | 1.42  |
| 80   | -3.67       | -3.22  | -2.60  | -2.11  | -1.59  | 0.878 | 1.08  | 1.24  | 1.39  | 1.48  |
| 90   | -3.61       | -3.17  | -2.57  | -2.09  | -1.58  | 0.902 | 1.12  | 1.28  | 1.44  | 1.54  |
| 100  | -3.57       | -3.14  | -2.54  | -2.07  | -1.57  | 0.923 | 1.14  | 1.31  | 1.48  | 1.59  |
| 150  | -3.409      | -3.009 | -2.452 | -2.004 | -1.520 | 0.990 | 1.233 | 1.423 | 1.623 | 1.746 |
| 200  | -3.302      | -2.922 | -2.391 | -1.960 | -1.491 | 1.032 | 1.290 | 1.496 | 1.715 | 1.853 |
| 250  | -3.227      | -2.861 | -2.348 | -1.926 | -1.471 | 1.060 | 1.328 | 1.545 | 1.779 | 1.927 |
| 300  | -3.172      | -2.816 | -2.316 | -1.906 | -1.456 | 1.080 | 1.357 | 1.528 | 1.826 | 1.983 |
| 350  | -3.129      | -2.781 | -2.291 | -1.888 | -1.444 | 1.096 | 1.379 | 1.610 | 1.863 | 2.026 |
| 400  | -3.094      | -2.753 | -2.270 | -1.873 | -1.434 | 1.108 | 1.396 | 1.633 | 1.893 | 2.061 |
| 450  | -3.064      | -2.729 | -2.253 | -1.861 | -1.426 | 1.119 | 1.411 | 1.652 | 1.918 | 2.090 |
| 500  | -3.040      | -2.709 | -2.239 | -1.850 | -1.419 | 1.127 | 1.423 | 1.668 | 1.938 | 2.114 |
| 550  | -3.019      | -2.691 | -2.226 | -1.841 | -1.413 | 1.135 | 1.434 | 1.682 | 1.957 | 2.136 |
| 600  | -3.000      | -2.676 | -2.215 | -1.833 | -1.408 | 1.141 | 1.443 | 1.694 | 1.972 | 2.154 |
| 650  | -2.984      | -2.663 | -2.206 | -1.826 | -1.403 | 1.147 | 1.451 | 1.704 | 1.986 | 2.171 |
| 700  | -2.969      | -2.651 | -2.197 | -1.820 | -1.399 | 1.152 | 1.458 | 1.714 | 1.999 | 2.185 |
| 750  | -2.956      | -2.640 | -2.189 | -1.814 | -1.395 | 1.157 | 1.465 | 1.722 | 2.010 | 2.199 |
| 800  | -2.944      | -2.630 | -2.182 | -1.809 | -1.392 | 1.161 | 1.471 | 1.730 | 2.020 | 2.221 |
| 850  | -2.933      | -2.621 | -2.176 | -1.804 | -1.389 | 1.165 | 1.476 | 1.737 | 2.029 | 2.221 |
| 900  | -2.923      | -2.613 | -2.710 | -1.800 | -1.386 | 1.168 | 1.481 | 1.743 | 2.037 | 2.231 |
| 950  | -2.914      | -2.605 | -2.164 | -1.796 | -1.383 | 1.171 | 1.485 | 1.749 | 2.045 | 2.241 |
| 1000 | -2.906      | -2.599 | -2.159 | -1.792 | -1.381 | 1.174 | 1.489 | 1.754 | 2.052 | 2.249 |
| 1500 | -2.845      | -2.549 | -2.123 | -1.765 | -1.363 | 1.194 | 1.519 | 1.793 | 2.103 | 2.309 |
| 2000 | -2.807      | -2.515 | -2.101 | -1.750 | -1.353 | 1.207 | 1.536 | 1.815 | 2.132 | 2.342 |

Table 4: Sample Data. Diameters at Breast Height (cm) of 584 Longleaf Pine Trees.  
Locations and Diameters at Breast Height (dbh, in centimeters) of all 584 Longleaf Pine Trees in the 4 hectare Study Region. The  $x$  coordinates are distances (in meters) from the tree to the southern boundary. The  $y$  coordinates are distances (in meters) from the tree to the eastern boundary. Reproduced from Table 8.1 of Cressie [9].

| $x$   | $y$   | dbh  | $x$   | $y$   | dbh  | $x$   | $y$   | dbh  | $x$   | $y$   | dbh  |
|-------|-------|------|-------|-------|------|-------|-------|------|-------|-------|------|
| 200.0 | 8.8   | 32.9 | 199.3 | 10.0  | 53.5 | 193.6 | 22.4  | 68.0 | 167.7 | 35.6  | 17.7 |
| 183.9 | 45.4  | 36.9 | 182.5 | 47.2  | 51.6 | 166.1 | 48.8  | 66.4 | 160.7 | 42.4  | 17.7 |
| 162.9 | 29.0  | 21.9 | 166.4 | 33.6  | 25.7 | 163.0 | 35.8  | 25.5 | 156.1 | 38.7  | 28.3 |
| 157.6 | 42.8  | 11.2 | 154.4 | 36.2  | 33.8 | 150.8 | 45.8  | 2.5  | 144.6 | 25.4  | 4.2  |
| 142.7 | 25.4  | 2.5  | 144.0 | 28.3  | 31.2 | 143.5 | 36.9  | 16.4 | 123.1 | 14.3  | 53.2 |
| 113.9 | 13.1  | 67.3 | 114.9 | 8.1   | 37.8 | 101.4 | 9.3   | 49.9 | 105.7 | 9.1   | 46.3 |
| 106.9 | 14.7  | 40.5 | 127.0 | 29.7  | 57.7 | 129.8 | 45.8  | 58.0 | 136.3 | 44.2  | 54.9 |
| 106.7 | 49.4  | 25.3 | 103.4 | 49.6  | 18.4 | 89.7  | 4.9   | 72.0 | 10.8  | 0.0   | 31.4 |
| 26.4  | 5.4   | 55.1 | 11.0  | 5.5   | 36.0 | 5.1   | 3.9   | 28.4 | 10.1  | 8.5   | 24.8 |
| 18.9  | 11.3  | 44.1 | 28.4  | 11.0  | 50.9 | 41.1  | 9.2   | 47.5 | 41.2  | 12.6  | 58.0 |
| 33.9  | 21.4  | 36.9 | 40.8  | 39.8  | 65.6 | 49.7  | 18.2  | 52.9 | 6.7   | 46.9  | 39.5 |
| 11.6  | 46.9  | 42.7 | 17.2  | 47.9  | 44.4 | 19.4  | 50.0  | 40.3 | 26.9  | 47.2  | 53.5 |
| 39.6  | 47.9  | 44.2 | 38.0  | 50.7  | 53.8 | 19.1  | 45.2  | 38.0 | 32.1  | 35.0  | 48.3 |
| 28.4  | 35.5  | 42.9 | 3.8   | 44.8  | 40.6 | 8.5   | 43.4  | 34.5 | 11.2  | 40.2  | 45.7 |
| 22.4  | 34.3  | 51.8 | 23.8  | 33.3  | 52.0 | 24.9  | 29.8  | 44.5 | 9.0   | 38.9  | 35.6 |
| 10.4  | 61.2  | 19.2 | 30.9  | 52.2  | 43.5 | 48.9  | 67.8  | 33.7 | 49.5  | 73.8  | 43.3 |
| 46.3  | 80.9  | 36.6 | 44.1  | 78.0  | 46.3 | 48.5  | 94.8  | 48.3 | 45.9  | 90.4  | 20.4 |
| 44.2  | 84.0  | 40.5 | 37.0  | 64.3  | 44.0 | 36.3  | 67.7  | 40.9 | 36.7  | 71.5  | 51.0 |
| 35.3  | 78.3  | 36.5 | 33.5  | 81.6  | 42.1 | 29.3  | 83.8  | 15.6 | 22.4  | 84.1  | 18.5 |
| 17.1  | 84.7  | 43.0 | 27.3  | 89.4  | 28.9 | 27.9  | 90.6  | 21.3 | 48.4  | 99.5  | 30.9 |
| 43.6  | 98.4  | 42.7 | 39.0  | 97.3  | 37.6 | 14.9  | 91.2  | 47.1 | 6.1   | 96.2  | 44.6 |
| 10.7  | 98.6  | 44.3 | 22.2  | 100.0 | 26.1 | 32.7  | 99.1  | 25.9 | 0.9   | 100.0 | 41.4 |
| 93.5  | 96.2  | 59.5 | 85.1  | 90.6  | 26.1 | 92.8  | 61.5  | 11.4 | 91.3  | 69.5  | 33.4 |
| 95.9  | 59.7  | 35.8 | 93.4  | 71.5  | 54.4 | 89.6  | 86.3  | 33.6 | 99.5  | 78.9  | 35.5 |
| 100.6 | 53.1  | 7.4  | 103.5 | 72.1  | 36.6 | 104.7 | 74.0  | 19.1 | 104.0 | 67.1  | 34.9 |
| 104.2 | 64.7  | 37.3 | 105.0 | 59.8  | 16.3 | 111.8 | 73.2  | 39.1 | 112.4 | 69.8  | 36.5 |
| 110.0 | 65.9  | 25.0 | 120.4 | 79.2  | 46.8 | 109.4 | 62.5  | 18.7 | 109.7 | 62.9  | 23.2 |
| 113.3 | 60.4  | 20.4 | 118.0 | 69.3  | 42.3 | 126.5 | 69.2  | 38.1 | 125.1 | 68.2  | 17.9 |
| 114.2 | 54.6  | 39.7 | 110.6 | 51.5  | 14.5 | 147.3 | 73.8  | 33.5 | 146.7 | 73.0  | 56.0 |
| 148.1 | 86.2  | 66.1 | 138.2 | 73.4  | 26.3 | 135.7 | 70.7  | 44.8 | 134.9 | 72.7  | 24.2 |
| 98.0  | 27.7  | 39.0 | 93.5  | 28.7  | 15.1 | 82.3  | 16.8  | 35.6 | 79.2  | 25.3  | 21.6 |
| 84.2  | 29.0  | 17.2 | 88.8  | 35.1  | 22.3 | 82.5  | 36.3  | 18.2 | 75.6  | 28.1  | 55.6 |
| 72.9  | 36.2  | 23.2 | 79.1  | 43.6  | 27.0 | 50.0  | 48.8  | 50.1 | 59.9  | 34.4  | 45.5 |
| 60.5  | 13.0  | 47.2 | 60.2  | 11.4  | 37.8 | 66.5  | 15.9  | 31.9 | 70.4  | 6.6   | 38.5 |
| 70.7  | 2.2   | 23.8 | 71.7  | 1.9   | 46.3 | 179.5 | 92.6  | 2.8  | 186.1 | 91.0  | 3.2  |
| 178.3 | 92.4  | 5.8  | 178.6 | 91.8  | 3.5  | 186.2 | 90.3  | 2.3  | 185.2 | 89.9  | 3.8  |
| 185.5 | 89.8  | 3.2  | 185.8 | 89.1  | 4.4  | 186.5 | 88.8  | 3.9  | 176.7 | 92.3  | 7.8  |
| 177.7 | 91.5  | 4.7  | 184.0 | 89.0  | 4.8  | 11.0  | 34.4  | 44.1 | 17.5  | 21.9  | 51.5 |
| 4.3   | 31.3  | 51.6 | 5.9   | 8.1   | 33.3 | 1.9   | 68.5  | 13.3 | 1.8   | 71.0  | 5.7  |
| 1.1   | 82.5  | 3.3  | 2.4   | 95.3  | 45.9 | 4.6   | 94.0  | 32.6 | 3.1   | 79.5  | 11.4 |
| 3.9   | 72.1  | 9.1  | 4.1   | 70.9  | 5.2  | 7.9   | 68.7  | 4.9  | 14.8  | 81.8  | 42.0 |
| 9.4   | 67.7  | 32.0 | 15.9  | 78.7  | 32.8 | 16.6  | 78.8  | 22.0 | 18.2  | 80.3  | 20.8 |
| 174.1 | 135.6 | 7.3  | 173.0 | 127.4 | 3.0  | 174.0 | 125.7 | 2.2  | 177.3 | 121.0 | 2.2  |
| 177.6 | 120.3 | 2.2  | 195.7 | 144.1 | 59.4 | 197.0 | 142.5 | 48.1 | 178.2 | 112.6 | 51.5 |
| 173.8 | 112.7 | 50.3 | 172.8 | 124.4 | 2.9  | 162.7 | 114.6 | 19.1 | 164.6 | 120.9 | 15.1 |
| 80.4  | 90.7  | 21.7 | 71.0  | 88.8  | 42.4 | 73.0  | 85.6  | 40.2 | 56.7  | 95.3  | 37.4 |
| 66.5  | 86.2  | 40.1 | 67.0  | 84.7  | 39.5 | 62.9  | 87.9  | 32.5 | 61.8  | 89.0  | 39.5 |
| 51.9  | 94.5  | 35.6 | 60.9  | 71.6  | 44.1 | 61.0  | 69.8  | 42.2 | 61.7  | 66.2  | 39.4 |
| 57.3  | 68.4  | 35.5 | 54.2  | 76.4  | 39.1 | 76.1  | 52.9  | 9.5  | 67.2  | 57.6  | 48.4 |
| 81.9  | 58.5  | 31.9 | 90.1  | 59.6  | 30.7 | 135.3 | 126.6 | 15.0 | 135.0 | 124.0 | 24.5 |

Table 4(continued).

| $x$   | $y$   | dbh  | $x$   | $y$   | dbh  | $x$   | $y$   | dbh  | $x$   | $y$   | dbh  |
|-------|-------|------|-------|-------|------|-------|-------|------|-------|-------|------|
| 136.2 | 122.1 | 15.0 | 129.7 | 127.0 | 22.2 | 134.8 | 120.2 | 27.5 | 136.9 | 116.8 | 10.8 |
| 137.0 | 116.0 | 26.2 | 128.9 | 124.2 | 10.2 | 127.5 | 125.0 | 18.9 | 127.6 | 121.7 | 44.2 |
| 129.7 | 119.0 | 13.8 | 126.6 | 121.1 | 16.7 | 133.4 | 77.1  | 35.7 | 129.9 | 76.1  | 12.1 |
| 126.5 | 77.3  | 35.4 | 129.1 | 83.1  | 32.7 | 134.4 | 87.0  | 30.1 | 130.7 | 90.1  | 28.4 |
| 130.9 | 90.7  | 16.5 | 132.0 | 94.5  | 12.7 | 136.8 | 96.7  | 5.5  | 137.7 | 98.0  | 2.5  |
| 157.8 | 99.9  | 3.0  | 187.1 | 98.1  | 3.2  | 190.6 | 92.1  | 3.2  | 185.4 | 93.1  | 4.0  |
| 186.6 | 92.2  | 3.6  | 185.9 | 91.7  | 3.8  | 184.3 | 92.1  | 4.3  | 188.2 | 91.2  | 3.3  |
| 104.4 | 145.1 | 6.3  | 104.9 | 145.0 | 18.4 | 101.5 | 148.4 | 5.4  | 102.4 | 148.7 | 5.4  |
| 123.4 | 128.9 | 26.0 | 123.8 | 135.1 | 22.3 | 127.0 | 133.8 | 35.2 | 109.6 | 145.9 | 24.1 |
| 112.4 | 145.0 | 6.9  | 133.1 | 144.8 | 61.0 | 139.4 | 143.1 | 20.6 | 140.4 | 143.6 | 6.5  |
| 184.1 | 88.2  | 2.8  | 183.5 | 88.5  | 4.8  | 183.0 | 88.0  | 5.4  | 176.1 | 91.0  | 4.3  |
| 175.6 | 90.2  | 4.0  | 173.8 | 89.9  | 3.2  | 164.9 | 93.7  | 2.8  | 163.0 | 95.3  | 4.9  |
| 163.2 | 94.1  | 3.5  | 162.4 | 94.5  | 2.9  | 161.5 | 94.9  | 2.4  | 162.2 | 94.3  | 3.3  |
| 161.0 | 94.7  | 2.1  | 157.7 | 95.7  | 2.0  | 154.9 | 96.2  | 3.9  | 154.6 | 92.7  | 5.0  |
| 152.9 | 93.7  | 2.3  | 153.2 | 93.2  | 2.2  | 168.2 | 73.0  | 67.7 | 151.6 | 93.0  | 2.9  |
| 151.4 | 93.4  | 2.4  | 157.6 | 67.2  | 56.3 | 149.4 | 63.0  | 39.4 | 149.4 | 64.3  | 59.5 |
| 167.3 | 54.6  | 42.4 | 157.4 | 51.5  | 63.7 | 181.5 | 66.1  | 66.6 | 196.5 | 55.2  | 69.3 |
| 189.9 | 85.2  | 56.9 | 155.1 | 149.2 | 23.5 | 154.5 | 148.4 | 9.1  | 162.9 | 119.9 | 29.9 |
| 158.4 | 113.4 | 14.9 | 153.9 | 108.3 | 38.7 | 156.1 | 116.0 | 31.5 | 156.5 | 118.9 | 27.8 |
| 156.8 | 122.3 | 28.5 | 159.0 | 126.1 | 21.6 | 161.0 | 131.9 | 2.0  | 161.3 | 132.8 | 2.6  |
| 160.6 | 132.6 | 2.3  | 161.3 | 134.9 | 3.5  | 159.7 | 129.8 | 3.6  | 161.7 | 136.1 | 2.6  |
| 161.1 | 136.4 | 2.0  | 160.1 | 133.0 | 2.0  | 159.0 | 133.6 | 2.7  | 160.0 | 134.8 | 2.6  |
| 160.2 | 135.5 | 2.2  | 159.1 | 136.5 | 2.7  | 154.7 | 126.8 | 30.1 | 151.9 | 127.5 | 16.6 |
| 151.3 | 124.7 | 10.4 | 151.0 | 127.3 | 11.8 | 150.4 | 123.0 | 32.3 | 149.6 | 124.6 | 33.5 |
| 146.2 | 127.1 | 30.5 | 146.1 | 127.4 | 10.5 | 144.4 | 131.8 | 13.8 | 143.3 | 131.5 | 22.8 |
| 140.6 | 137.7 | 31.7 | 143.2 | 125.4 | 10.1 | 127.1 | 119.9 | 14.5 | 120.7 | 115.6 | 12.0 |
| 115.3 | 112.6 | 2.2  | 134.1 | 105.2 | 2.3  | 134.6 | 104.1 | 3.2  | 135.6 | 103.3 | 3.0  |
| 128.9 | 102.6 | 50.6 | 116.3 | 106.5 | 2.6  | 104.3 | 104.0 | 50.0 | 111.5 | 100.0 | 52.2 |
| 100.5 | 149.7 | 5.2  | 100.0 | 145.5 | 5.2  | 100.8 | 145.0 | 6.7  | 100.9 | 143.5 | 14.0 |
| 100.3 | 140.8 | 12.7 | 101.5 | 120.8 | 59.5 | 99.3  | 110.6 | 52.0 | 99.2  | 106.0 | 45.9 |
| 102.0 | 137.1 | 18.0 | 105.4 | 115.7 | 43.5 | 103.6 | 134.2 | 3.3  | 103.9 | 139.4 | 4.3  |
| 102.6 | 141.6 | 7.4  | 102.0 | 143.3 | 10.1 | 102.1 | 144.4 | 23.1 | 103.5 | 141.3 | 8.1  |
| 102.9 | 143.8 | 5.7  | 105.7 | 138.2 | 13.3 | 106.6 | 135.1 | 12.8 | 108.5 | 133.2 | 11.6 |
| 105.2 | 142.3 | 6.3  | 139.7 | 145.8 | 20.0 | 145.5 | 148.4 | 8.9  | 146.4 | 148.4 | 27.6 |
| 105.8 | 149.8 | 4.5  | 96.7  | 149.1 | 9.2  | 66.5  | 150.0 | 2.3  | 55.7  | 148.5 | 5.0  |
| 54.7  | 146.8 | 4.0  | 57.1  | 144.0 | 21.8 | 61.7  | 145.3 | 10.9 | 60.1  | 143.7 | 14.9 |
| 77.7  | 144.8 | 45.0 | 67.2  | 139.3 | 16.4 | 80.7  | 133.2 | 43.3 | 85.1  | 133.5 | 55.6 |
| 94.7  | 143.7 | 10.6 | 81.2  | 125.0 | 45.9 | 81.9  | 123.2 | 45.2 | 83.8  | 123.1 | 35.5 |
| 84.8  | 121.4 | 43.6 | 82.9  | 119.2 | 44.6 | 82.1  | 116.4 | 38.8 | 84.3  | 114.8 | 34.9 |
| 96.7  | 142.6 | 17.0 | 92.0  | 109.0 | 50.4 | 96.1  | 146.6 | 2.0  | 78.5  | 102.5 | 33.8 |
| 78.7  | 103.0 | 51.1 | 59.5  | 107.4 | 21.8 | 56.5  | 105.5 | 46.5 | 64.3  | 132.1 | 5.6  |
| 152.7 | 146.7 | 19.6 | 155.8 | 145.4 | 32.3 | 161.2 | 138.1 | 3.7  | 161.0 | 138.1 | 2.7  |
| 162.1 | 136.9 | 2.5  | 166.2 | 132.0 | 2.5  | 168.7 | 133.4 | 2.4  | 169.3 | 133.7 | 7.2  |
| 57.9  | 140.7 | 7.0  | 57.5  | 142.3 | 11.8 | 57.3  | 141.7 | 8.5  | 56.0  | 137.7 | 9.5  |
| 53.4  | 139.3 | 7.0  | 53.1  | 136.0 | 10.5 | 54.0  | 137.7 | 6.6  | 54.5  | 136.7 | 6.6  |
| 53.3  | 137.8 | 8.8  | 52.1  | 139.3 | 11.6 | 48.0  | 114.4 | 48.2 | 44.2  | 129.6 | 36.2 |
| 39.4  | 136.8 | 44.9 | 42.7  | 124.0 | 43.0 | 38.1  | 134.4 | 37.5 | 37.1  | 131.9 | 31.5 |
| 37.6  | 125.4 | 39.9 | 31.2  | 127.9 | 35.5 | 40.1  | 112.2 | 51.7 | 29.3  | 118.6 | 36.5 |
| 23.8  | 114.5 | 40.2 | 141.0 | 127.8 | 7.8  | 140.1 | 127.3 | 17.0 | 140.9 | 121.4 | 36.4 |
| 135.0 | 132.3 | 19.6 | 139.3 | 122.9 | 15.0 | 142.0 | 117.2 | 28.8 | 140.4 | 117.2 | 20.1 |

Table 4(continued).

| $x$   | $y$   | dbh  | $x$   | $y$   | dbh  | $x$   | $y$   | dbh  | $x$   | $y$   | dbh  |
|-------|-------|------|-------|-------|------|-------|-------|------|-------|-------|------|
| 138.5 | 121.5 | 39.3 | 28.7  | 158.8 | 37.9 | 33.7  | 162.3 | 40.6 | 23.1  | 160.8 | 33.0 |
| 11.3  | 158.9 | 35.7 | 18.2  | 168.2 | 20.6 | 21.5  | 172.3 | 22.0 | 15.9  | 168.3 | 16.3 |
| 15.4  | 172.8 | 5.6  | 14.0  | 174.2 | 7.4  | 6.8   | 179.6 | 42.3 | 6.0   | 184.1 | 43.8 |
| 1.6   | 194.9 | 53.0 | 43.6  | 197.3 | 48.1 | 39.4  | 195.5 | 41.9 | 37.1  | 196.1 | 48.0 |
| 23.7  | 193.9 | 75.9 | 21.5  | 187.9 | 40.4 | 27.7  | 188.7 | 40.9 | 32.3  | 178.9 | 39.4 |
| 32.6  | 168.6 | 40.9 | 37.7  | 176.9 | 17.6 | 107.5 | 138.5 | 17.8 | 107.9 | 139.5 | 3.7  |
| 116.5 | 122.6 | 19.0 | 114.5 | 127.7 | 11.2 | 115.3 | 127.4 | 27.6 | 115.3 | 128.1 | 14.5 |
| 119.0 | 127.4 | 34.4 | 119.4 | 127.7 | 20.0 | 94.7  | 179.8 | 2.9  | 89.3  | 185.0 | 7.3  |
| 90.8  | 174.0 | 52.7 | 95.3  | 158.4 | 8.7  | 90.9  | 162.1 | 3.6  | 90.2  | 162.1 | 4.6  |
| 90.2  | 161.7 | 11.4 | 90.6  | 160.8 | 11.0 | 93.0  | 158.0 | 18.7 | 78.4  | 172.4 | 5.6  |
| 76.2  | 171.4 | 2.1  | 75.8  | 171.0 | 3.3  | 75.7  | 169.7 | 11.5 | 82.7  | 163.5 | 2.6  |
| 76.7  | 166.3 | 4.4  | 74.7  | 167.1 | 18.3 | 119.4 | 170.8 | 7.5  | 74.2  | 164.3 | 17.2 |
| 73.9  | 162.7 | 4.6  | 81.7  | 156.7 | 32.0 | 79.5  | 156.3 | 56.7 | 56.8  | 116.0 | 46.0 |
| 62.2  | 137.7 | 7.8  | 58.2  | 125.1 | 54.9 | 54.1  | 115.5 | 45.5 | 59.5  | 138.1 | 9.2  |
| 58.6  | 140.3 | 13.2 | 58.8  | 141.5 | 15.3 | 57.9  | 137.3 | 8.5  | 153.5 | 159.9 | 2.2  |
| 155.9 | 183.7 | 58.8 | 160.4 | 176.6 | 47.5 | 171.3 | 185.1 | 52.2 | 182.8 | 187.4 | 56.3 |
| 182.5 | 196.0 | 39.8 | 176.3 | 197.7 | 38.1 | 161.9 | 199.4 | 38.9 | 199.5 | 179.4 | 9.7  |
| 197.6 | 176.9 | 7.4  | 196.3 | 192.4 | 22.1 | 195.7 | 180.5 | 16.9 | 196.2 | 177.1 | 5.9  |
| 196.3 | 176.0 | 10.5 | 193.7 | 185.8 | 9.5  | 191.7 | 189.2 | 45.9 | 194.5 | 173.8 | 11.4 |
| 192.7 | 177.3 | 7.8  | 188.9 | 182.1 | 14.4 | 190.1 | 174.4 | 8.3  | 186.9 | 179.4 | 30.6 |
| 26.9  | 111.3 | 44.4 | 17.9  | 111.0 | 38.7 | 34.4  | 104.2 | 41.5 | 31.9  | 103.2 | 34.5 |
| 20.6  | 101.5 | 31.8 | 14.1  | 103.1 | 39.7 | 2.9   | 122.8 | 23.3 | 6.4   | 125.9 | 37.7 |
| 2.2   | 142.2 | 43.0 | 11.7  | 116.2 | 39.2 | 14.2  | 116.5 | 40.4 | 15.6  | 118.1 | 36.7 |
| 13.6  | 127.4 | 48.4 | 11.1  | 134.8 | 27.9 | 7.2   | 141.7 | 46.4 | 12.2  | 140.1 | 38.5 |
| 23.0  | 132.7 | 39.4 | 30.2  | 133.9 | 50.0 | 27.7  | 136.5 | 51.6 | 3.4   | 148.8 | 38.7 |
| 15.4  | 145.6 | 39.6 | 16.7  | 146.4 | 29.1 | 24.3  | 145.7 | 44.0 | 0.4   | 175.2 | 50.9 |
| 0.0   | 177.5 | 50.8 | 7.9   | 151.0 | 43.0 | 33.2  | 151.2 | 44.5 | 36.6  | 150.6 | 29.8 |
| 42.2  | 153.7 | 44.3 | 24.5  | 153.4 | 51.2 | 40.4  | 179.3 | 37.7 | 41.0  | 176.6 | 36.8 |
| 43.9  | 182.2 | 33.6 | 44.7  | 184.6 | 47.9 | 45.6  | 175.2 | 32.0 | 47.5  | 175.9 | 40.3 |
| 51.2  | 177.9 | 42.5 | 55.0  | 159.3 | 59.7 | 58.0  | 180.3 | 44.2 | 54.6  | 188.7 | 30.9 |
| 58.9  | 180.0 | 39.5 | 63.9  | 178.6 | 48.7 | 64.3  | 178.9 | 32.8 | 65.6  | 179.3 | 47.2 |
| 61.0  | 184.9 | 42.1 | 63.1  | 183.3 | 43.8 | 86.1  | 186.9 | 30.5 | 65.8  | 194.9 | 28.3 |
| 90.0  | 195.1 | 10.4 | 94.3  | 196.1 | 15.0 | 91.9  | 197.1 | 7.4  | 86.5  | 197.4 | 15.3 |
| 87.5  | 199.3 | 17.5 | 93.9  | 199.2 | 5.0  | 92.4  | 199.3 | 12.2 | 81.8  | 198.9 | 9.0  |
| 99.0  | 158.1 | 2.4  | 94.1  | 187.2 | 13.7 | 95.4  | 182.9 | 13.1 | 97.1  | 168.4 | 12.8 |
| 79.2  | 155.6 | 27.0 | 61.6  | 158.2 | 2.6  | 70.3  | 153.1 | 4.9  | 79.8  | 151.8 | 35.0 |
| 110.1 | 150.4 | 23.7 | 116.1 | 156.8 | 42.9 | 114.0 | 165.1 | 14.2 | 103.2 | 154.4 | 3.3  |
| 112.3 | 167.0 | 28.4 | 110.4 | 167.3 | 10.0 | 110.6 | 166.4 | 6.4  | 107.0 | 165.0 | 22.0 |
| 105.6 | 160.6 | 4.3  | 104.0 | 162.4 | 10.0 | 104.0 | 166.1 | 9.2  | 103.7 | 167.2 | 3.7  |
| 108.6 | 182.1 | 66.7 | 105.7 | 182.6 | 68.0 | 102.8 | 169.7 | 23.1 | 101.5 | 171.8 | 5.7  |
| 100.4 | 170.5 | 11.7 | 144.1 | 199.0 | 40.4 | 138.3 | 197.9 | 43.3 | 142.7 | 197.2 | 60.2 |
| 118.8 | 188.0 | 55.5 | 142.3 | 173.3 | 54.1 | 143.8 | 156.0 | 22.3 | 145.3 | 155.6 | 21.4 |
| 151.2 | 192.2 | 55.7 | 153.7 | 176.5 | 51.4 | 186.9 | 174.7 | 23.9 | 181.2 | 176.9 | 5.2  |
| 181.1 | 176.1 | 7.6  | 177.2 | 174.5 | 27.8 | 182.8 | 162.9 | 49.6 | 180.0 | 160.2 | 51.0 |
| 189.1 | 156.3 | 50.7 | 196.9 | 151.4 | 43.4 | 171.4 | 161.6 | 55.6 | 169.1 | 160.0 | 4.3  |
| 162.5 | 157.3 | 2.5  | 156.7 | 155.3 | 23.5 | 154.1 | 150.8 | 8.0  | 87.7  | 200.0 | 11.7 |

Table 5: Coefficients for transforming  $\sqrt{b_1}$  to a standard normal using a Johnson  $S_U$  approximation.

Reproduced from Table 4 of D'Agostino and Pearson [11].

| $n$ | $\delta$ | $1/\lambda$ | $n$ | $\delta$ | $1/\lambda$ | $n$  | $\delta$ | $1/\lambda$ |
|-----|----------|-------------|-----|----------|-------------|------|----------|-------------|
| 8   | 5.563    | 0.3030      | 62  | 3.389    | 1.0400      | 260  | 5.757    | 1.1744      |
| 9   | 4.260    | 0.4080      | 64  | 3.420    | 1.0449      | 270  | 5.835    | 1.1761      |
| 10  | 3.734    | 0.4794      | 66  | 3.450    | 1.0495      | 280  | 5.946    | 1.1779      |
|     |          |             | 68  | 3.480    | 1.0540      | 290  | 6.039    | 1.1793      |
| 11  | 3.447    | 0.5339      | 70  | 3.510    | 1.0581      | 300  | 6.130    | 1.1808      |
| 12  | 3.270    | 0.5781      |     |          |             |      |          |             |
| 13  | 3.151    | 0.6153      | 72  | 3.540    | 1.0621      | 310  | 6.220    | 1.1821      |
| 14  | 3.069    | 0.6473      | 74  | 3.569    | 1.0659      | 320  | 6.308    | 1.1834      |
| 15  | 3.010    | 0.6753      | 76  | 3.599    | 1.0695      | 330  | 6.396    | 1.1846      |
|     |          |             | 78  | 3.628    | 1.0730      | 340  | 6.482    | 1.1858      |
| 16  | 2.968    | 0.7001      | 80  | 3.657    | 1.0763      | 350  | 6.567    | 1.1868      |
| 17  | 2.937    | 0.7224      |     |          |             |      |          |             |
| 18  | 2.915    | 0.7426      | 82  | 3.686    | 1.0795      | 360  | 6.651    | 1.1879      |
| 19  | 2.900    | 0.7610      | 84  | 3.715    | 1.0825      | 370  | 6.733    | 1.1888      |
| 20  | 2.890    | 0.7779      | 86  | 3.744    | 1.0854      | 380  | 6.815    | 1.1897      |
|     |          |             | 88  | 3.772    | 1.0882      | 390  | 6.896    | 1.1906      |
| 21  | 2.884    | 0.7934      | 90  | 3.801    | 1.0909      | 400  | 6.976    | 1.1914      |
| 22  | 2.882    | 0.8078      |     |          |             |      |          |             |
| 23  | 2.882    | 0.8211      | 92  | 3.829    | 1.0934      | 410  | 7.056    | 1.1922      |
| 24  | 2.884    | 0.8336      | 94  | 3.857    | 1.0959      | 420  | 7.134    | 1.1929      |
| 25  | 2.889    | 0.8452      | 96  | 3.885    | 1.0983      | 430  | 7.211    | 1.1937      |
|     |          |             | 98  | 3.913    | 1.1006      | 440  | 7.288    | 1.1943      |
| 26  | 2.895    | 0.8561      | 100 | 3.940    | 1.1028      | 450  | 7.363    | 1.1950      |
| 27  | 2.902    | 0.8664      |     |          |             |      |          |             |
| 28  | 2.910    | 0.8760      | 105 | 4.009    | 1.1080      | 460  | 7.438    | 1.1956      |
| 29  | 2.920    | 0.8851      | 110 | 4.076    | 1.1128      | 470  | 7.513    | 1.1962      |
| 30  | 2.930    | 0.8938      | 115 | 4.142    | 1.1172      | 480  | 7.586    | 1.1968      |
|     |          |             | 120 | 4.207    | 1.1212      | 490  | 7.659    | 1.1974      |
| 31  | 2.941    | 0.9020      | 125 | 4.272    | 1.1250      | 500  | 7.731    | 1.1959      |
| 32  | 2.952    | 0.9097      |     |          |             |      |          |             |
| 33  | 2.964    | 0.9171      | 130 | 4.336    | 1.1285      | 520  | 7.873    | 1.1989      |
| 34  | 2.977    | 0.9241      | 135 | 4.398    | 1.1318      | 540  | 8.013    | 1.1998      |
| 35  | 2.990    | 0.9308      | 140 | 4.460    | 1.1348      | 560  | 8.151    | 1.2007      |
|     |          |             | 145 | 4.521    | 1.1377      | 580  | 8.286    | 1.2015      |
| 36  | 3.003    | 0.9372      | 150 | 4.582    | 1.1403      | 600  | 8.419    | 1.2023      |
| 37  | 3.016    | 0.9433      |     |          |             |      |          |             |
| 38  | 3.030    | 0.9492      | 155 | 4.641    | 1.1428      | 620  | 8.550    | 1.2030      |
| 39  | 3.044    | 0.9548      | 160 | 4.700    | 1.1452      | 640  | 8.679    | 1.2036      |
| 40  | 3.058    | 0.9601      | 165 | 4.758    | 1.1474      | 660  | 8.806    | 1.2043      |
|     |          |             | 170 | 4.816    | 1.1496      | 680  | 8.931    | 1.2049      |
| 41  | 3.073    | 0.9653      | 175 | 4.873    | 1.1516      | 700  | 9.054    | 1.2054      |
| 42  | 3.087    | 0.9702      |     |          |             |      |          |             |
| 43  | 3.102    | 0.9750      | 180 | 4.929    | 1.1535      | 720  | 9.176    | 1.2060      |
| 44  | 3.117    | 0.9795      | 185 | 4.985    | 1.1553      | 740  | 9.297    | 1.2065      |
| 45  | 3.131    | 0.9840      | 190 | 5.040    | 1.1570      | 760  | 9.415    | 1.2069      |
|     |          |             | 195 | 5.094    | 1.1586      | 780  | 9.533    | 1.2073      |
| 46  | 3.146    | 0.9882      | 200 | 5.148    | 1.1602      | 800  | 9.649    | 1.2078      |
| 47  | 3.161    | 0.9923      |     |          |             |      |          |             |
| 48  | 3.176    | 0.9963      | 205 | 5.202    | 1.1616      | 820  | 9.763    | 1.2082      |
| 49  | 3.192    | 1.0001      | 210 | 5.255    | 1.1631      | 840  | 9.876    | 1.2086      |
| 50  | 3.207    | 1.0038      | 215 | 5.307    | 1.1644      | 860  | 9.988    | 1.2089      |
|     |          |             | 220 | 5.359    | 1.1657      | 880  | 10.098   | 1.2093      |
| 52  | 3.237    | 1.0108      | 225 | 5.410    | 1.1669      | 900  | 10.208   | 1.2096      |
| 54  | 3.268    | 1.0174      |     |          |             |      |          |             |
| 56  | 3.298    | 1.0235      | 230 | 5.461    | 1.1681      | 920  | 10.316   | 1.2100      |
| 58  | 3.329    | 1.0293      | 235 | 5.511    | 1.1693      | 940  | 10.423   | 1.2103      |
| 60  | 3.359    | 1.0348      | 240 | 5.561    | 1.1704      | 960  | 10.529   | 1.2106      |
|     |          |             | 245 | 5.611    | 1.1714      | 980  | 10.634   | 1.2109      |
|     |          |             | 250 | 5.660    | 1.1724      | 1000 | 10.738   | 1.2111      |

Table 6: Coefficients  $\{a_{n-i+1}\}$  for the Shapiro-Wilk  $W$  Test for Normality. Reproduced from Table 5 of Shapiro and Wilk [32].

| $i$ | $n$    |        |        |        |        |        |        |        |        |        |
|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
|     | 2      | 3      | 4      | 5      | 6      | 7      | 8      | 9      | 10     |        |
| 1   | 0.7071 | 0.7071 | 0.6872 | 0.6646 | 0.6431 | 0.6233 | 0.6052 | 0.5888 | 0.5739 |        |
| 2   | -      | 0.0000 | 0.1677 | 0.2413 | 0.2806 | 0.3031 | 0.3164 | 0.3244 | 0.3291 |        |
| 3   | -      | -      | -      | 0.0000 | 0.0875 | 0.1401 | 0.1743 | 0.1976 | 0.2141 |        |
| 4   | -      | -      | -      | -      | -      | 0.0000 | 0.0561 | 0.0947 | 0.1224 |        |
| 5   | -      | -      | -      | -      | -      | -      | -      | 0.0000 | 0.0399 |        |
|     | 11     | 12     | 13     | 14     | 15     | 16     | 17     | 18     | 19     | 20     |
| 1   | 0.5601 | 0.5475 | 0.5359 | 0.5251 | 0.5150 | 0.5056 | 0.4968 | 0.4886 | 0.4808 | 0.4734 |
| 2   | 0.3315 | 0.3325 | 0.3325 | 0.3318 | 0.3306 | 0.3290 | 0.3273 | 0.3253 | 0.3232 | 0.3211 |
| 3   | 0.2260 | 0.2347 | 0.2412 | 0.2460 | 0.2495 | 0.2521 | 0.2540 | 0.2553 | 0.2561 | 0.2565 |
| 4   | 0.1429 | 0.1586 | 0.1707 | 0.1802 | 0.1878 | 0.1939 | 0.1988 | 0.2027 | 0.2059 | 0.2085 |
| 5   | 0.0695 | 0.0922 | 0.1099 | 0.1240 | 0.1353 | 0.1447 | 0.1524 | 0.1587 | 0.1641 | 0.1686 |
| 6   | 0.0000 | 0.0303 | 0.0539 | 0.0727 | 0.0880 | 0.1005 | 0.1109 | 0.1197 | 0.1271 | 0.1334 |
| 7   | -      | -      | 0.0000 | 0.0240 | 0.0433 | 0.0593 | 0.0725 | 0.0837 | 0.0932 | 0.1013 |
| 8   | -      | -      | -      | -      | 0.0000 | 0.0196 | 0.0359 | 0.0496 | 0.0612 | 0.0711 |
| 9   | -      | -      | -      | -      | -      | -      | 0.0000 | 0.0163 | 0.0303 | 0.0422 |
| 10  | -      | -      | -      | -      | -      | -      | -      | -      | 0.0000 | 0.0140 |
|     | 21     | 22     | 23     | 24     | 25     | 26     | 27     | 28     | 29     | 30     |
| 1   | 0.4643 | 0.4590 | 0.4542 | 0.4493 | 0.4450 | 0.4407 | 0.4366 | 0.4328 | 0.4291 | 0.4254 |
| 2   | 0.3185 | 0.3156 | 0.3126 | 0.3098 | 0.3069 | 0.3043 | 0.3018 | 0.2992 | 0.2968 | 0.2944 |
| 3   | 0.2578 | 0.2571 | 0.2563 | 0.2554 | 0.2543 | 0.2533 | 0.2522 | 0.2510 | 0.2499 | 0.2487 |
| 4   | 0.2119 | 0.2131 | 0.2139 | 0.2145 | 0.2148 | 0.2151 | 0.2152 | 0.2151 | 0.2150 | 0.2148 |
| 5   | 0.1736 | 0.1764 | 0.1787 | 0.1807 | 0.1822 | 0.1836 | 0.1848 | 0.1857 | 0.1864 | 0.1870 |
| 6   | 0.1399 | 0.1443 | 0.1480 | 0.1512 | 0.1539 | 0.1563 | 0.1584 | 0.1601 | 0.1616 | 0.1630 |
| 7   | 0.1092 | 0.1150 | 0.1201 | 0.1245 | 0.1283 | 0.1316 | 0.1346 | 0.1372 | 0.1395 | 0.1415 |
| 8   | 0.0804 | 0.0878 | 0.0941 | 0.0997 | 0.1046 | 0.1089 | 0.1128 | 0.1162 | 0.1192 | 0.1219 |
| 9   | 0.0530 | 0.0618 | 0.0696 | 0.0764 | 0.0823 | 0.0876 | 0.0923 | 0.0965 | 0.1002 | 0.1036 |
| 10  | 0.0263 | 0.0368 | 0.0459 | 0.0539 | 0.0610 | 0.0672 | 0.0728 | 0.0778 | 0.0822 | 0.0862 |
| 11  | 0.0000 | 0.0122 | 0.0228 | 0.0321 | 0.0403 | 0.0476 | 0.0540 | 0.0598 | 0.0650 | 0.0697 |
| 12  | -      | -      | 0.0000 | 0.0107 | 0.0200 | 0.0284 | 0.0358 | 0.0424 | 0.0483 | 0.0537 |
| 13  | -      | -      | -      | -      | 0.0000 | 0.0094 | 0.0178 | 0.0253 | 0.0320 | 0.0381 |
| 14  | -      | -      | -      | -      | -      | -      | 0.0000 | 0.0084 | 0.0159 | 0.0227 |
| 15  | -      | -      | -      | -      | -      | -      | -      | -      | 0.0000 | 0.0076 |
|     | 31     | 32     | 33     | 34     | 35     | 36     | 37     | 38     | 39     | 40     |
| 1   | 0.4220 | 0.4188 | 0.4156 | 0.4127 | 0.4096 | 0.4068 | 0.4040 | 0.4015 | 0.3989 | 0.3964 |
| 2   | 0.2921 | 0.2898 | 0.2876 | 0.2854 | 0.2834 | 0.2813 | 0.2794 | 0.2774 | 0.2755 | 0.2737 |
| 3   | 0.2475 | 0.2463 | 0.2451 | 0.2439 | 0.2427 | 0.2415 | 0.2403 | 0.2391 | 0.2380 | 0.2368 |
| 4   | 0.2145 | 0.2141 | 0.2137 | 0.2132 | 0.2127 | 0.2121 | 0.2116 | 0.2110 | 0.2104 | 0.2098 |
| 5   | 0.1874 | 0.1878 | 0.1880 | 0.1882 | 0.1883 | 0.1883 | 0.1883 | 0.1881 | 0.1880 | 0.1878 |
| 6   | 0.1641 | 0.1651 | 0.1660 | 0.1667 | 0.1673 | 0.1678 | 0.1683 | 0.1686 | 0.1689 | 0.1691 |
| 7   | 0.1433 | 0.1449 | 0.1463 | 0.1475 | 0.1487 | 0.1496 | 0.1505 | 0.1513 | 0.1520 | 0.1526 |
| 8   | 0.1243 | 0.1265 | 0.1284 | 0.1301 | 0.1317 | 0.1331 | 0.1344 | 0.1356 | 0.1366 | 0.1376 |
| 9   | 0.1066 | 0.1093 | 0.1118 | 0.1140 | 0.1160 | 0.1179 | 0.1196 | 0.1211 | 0.1225 | 0.1237 |
| 10  | 0.0899 | 0.0931 | 0.0961 | 0.0988 | 0.1013 | 0.1036 | 0.1056 | 0.1075 | 0.1092 | 0.1108 |
| 11  | 0.0739 | 0.0777 | 0.0812 | 0.0844 | 0.0873 | 0.0900 | 0.0924 | 0.0947 | 0.0967 | 0.0986 |
| 12  | 0.0585 | 0.0629 | 0.0669 | 0.0706 | 0.0739 | 0.0770 | 0.0798 | 0.0824 | 0.0848 | 0.0870 |
| 13  | 0.0435 | 0.0485 | 0.0530 | 0.0572 | 0.0610 | 0.0645 | 0.0677 | 0.0706 | 0.0733 | 0.0759 |
| 14  | 0.0289 | 0.0344 | 0.0395 | 0.0441 | 0.0484 | 0.0523 | 0.0559 | 0.0592 | 0.0622 | 0.0651 |
| 15  | 0.0144 | 0.0206 | 0.0262 | 0.0314 | 0.0361 | 0.0404 | 0.0444 | 0.0481 | 0.0515 | 0.0546 |
| 16  | 0.0000 | 0.0068 | 0.0131 | 0.0187 | 0.0239 | 0.0287 | 0.0331 | 0.0372 | 0.0409 | 0.0444 |
| 17  | -      | -      | 0.0000 | 0.0062 | 0.0119 | 0.0172 | 0.0220 | 0.0264 | 0.0305 | 0.0343 |
| 18  | -      | -      | -      | -      | 0.0000 | 0.0057 | 0.0110 | 0.0158 | 0.0203 | 0.0244 |
| 19  | -      | -      | -      | -      | -      | -      | 0.0000 | 0.0053 | 0.0101 | 0.0146 |
| 20  | -      | -      | -      | -      | -      | -      | -      | -      | 0.0000 | 0.0049 |
|     | 41     | 42     | 43     | 44     | 45     | 46     | 47     | 48     | 49     | 50     |
| 1   | 0.3940 | 0.3917 | 0.3894 | 0.3872 | 0.3850 | 0.3830 | 0.3808 | 0.3789 | 0.3770 | 0.3964 |
| 2   | 0.2719 | 0.2701 | 0.2684 | 0.2667 | 0.2651 | 0.2635 | 0.2620 | 0.2604 | 0.2589 | 0.2737 |
| 3   | 0.2357 | 0.2345 | 0.2334 | 0.2323 | 0.2313 | 0.2302 | 0.2291 | 0.2281 | 0.2271 | 0.2368 |
| 4   | 0.2091 | 0.2085 | 0.2078 | 0.2072 | 0.2065 | 0.2058 | 0.2052 | 0.2045 | 0.2038 | 0.2098 |
| 5   | 0.1876 | 0.1874 | 0.1871 | 0.1868 | 0.1865 | 0.1862 | 0.1859 | 0.1855 | 0.1851 | 0.1878 |
| 6   | 0.1693 | 0.1694 | 0.1695 | 0.1695 | 0.1695 | 0.1695 | 0.1695 | 0.1693 | 0.1692 | 0.1691 |
| 7   | 0.1531 | 0.1535 | 0.1539 | 0.1542 | 0.1545 | 0.1548 | 0.1550 | 0.1551 | 0.1553 | 0.1554 |
| 8   | 0.1384 | 0.1392 | 0.1398 | 0.1405 | 0.1410 | 0.1415 | 0.1420 | 0.1423 | 0.1427 | 0.1430 |
| 9   | 0.1249 | 0.1259 | 0.1269 | 0.1278 | 0.1286 | 0.1293 | 0.1300 | 0.1306 | 0.1312 | 0.1317 |
| 10  | 0.1123 | 0.1136 | 0.1149 | 0.1160 | 0.1170 | 0.1180 | 0.1189 | 0.1197 | 0.1205 | 0.1212 |
| 11  | 0.1004 | 0.1020 | 0.1035 | 0.1049 | 0.1062 | 0.1073 | 0.1085 | 0.1095 | 0.1105 | 0.1113 |
| 12  | 0.0891 | 0.0909 | 0.0927 | 0.0943 | 0.0959 | 0.0972 | 0.0986 | 0.0998 | 0.1010 | 0.1020 |
| 13  | 0.0782 | 0.0804 | 0.0824 | 0.0842 | 0.0860 | 0.0876 | 0.0892 | 0.0906 | 0.0919 | 0.0932 |
| 14  | 0.0677 | 0.0701 | 0.0724 | 0.0745 | 0.0765 | 0.0783 | 0.0801 | 0.0817 | 0.0832 | 0.0846 |
| 15  | 0.0575 | 0.0602 | 0.0628 | 0.0651 | 0.0673 | 0.0694 | 0.0713 | 0.0731 | 0.0748 | 0.0764 |
| 16  | 0.0476 | 0.0506 | 0.0534 | 0.0560 | 0.0584 | 0.0607 | 0.0628 | 0.0648 | 0.0667 | 0.0685 |
| 17  | 0.0379 | 0.0411 | 0.0442 | 0.0471 | 0.0497 | 0.0522 | 0.0546 | 0.0568 | 0.0588 | 0.0608 |
| 18  | 0.0283 | 0.0318 | 0.0352 | 0.0383 | 0.0412 | 0.0439 | 0.0465 | 0.0489 | 0.0511 | 0.0532 |
| 19  | 0.0188 | 0.0227 | 0.0263 | 0.0296 | 0.0328 | 0.0357 | 0.0385 | 0.0411 | 0.0436 | 0.0459 |
| 20  | 0.0094 | 0.0136 | 0.0175 | 0.0211 | 0.0245 | 0.0277 | 0.0307 | 0.0335 | 0.0361 | 0.0386 |
| 21  | -      | 0.0045 | 0.0087 | 0.0126 | 0.0163 | 0.0197 | 0.0229 | 0.0259 | 0.0288 | 0.0314 |
| 22  | -      | -      | 0.0000 | 0.0042 | 0.0081 | 0.0118 | 0.0153 | 0.0185 | 0.0215 | 0.0244 |
| 23  | -      | -      | -      | -      | 0.0000 | 0.0039 | 0.0076 | 0.0111 | 0.0143 | 0.0174 |
| 24  | -      | -      | -      | -      | -      | -      | 0.0000 | 0.0037 | 0.0071 | 0.0104 |
| 25  | -      | -      | -      | -      | -      | -      | -      | -      | 0.0000 | 0.0035 |

Table 7: Critical Values of the Shapiro-Wilk  $W$  for Testing Normality.  
 Reproduced from Table 6 of Shapiro and Wilk [32].

| $n$ | $\alpha$ |       |       |       |       |
|-----|----------|-------|-------|-------|-------|
|     | 0.01     | 0.02  | 0.05  | 0.10  | 0.50  |
| 3   | 0.753    | 0.756 | 0.767 | 0.789 | 0.959 |



Table 8: Critical Values of the Shapiro-Wilk  $W$  for Testing Exponentiality.  
 Reproduced from Table 1 of Shapiro and Wilk [33].

| $n$ | $\alpha$ |       |       |       |       |       |       |       |       |       |        |
|-----|----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|--------|
|     | 0.005    | 0.01  | 0.025 | 0.05  | 0.10  | 0.50  | 0.90  | 0.95  | 0.975 | 0.99  | 0.995  |
| 3   | .2519    | .2538 | .2596 | .2697 | .2915 | .5714 | .9709 | .9926 | .9981 | .9997 | .99993 |
| 4   | .1241    | .1302 | .1434 | .1604 | .1891 | .3768 | .7514 | .8581 | .9236 | .9680 | .9837  |

Table 9: Coefficients  $\{b_{n-i+1}\}$  for the Shapiro-Francia  $W'$  Test for Normality.  
Reproduced from Table 1 of Shapiro and Wilk [31].

Table 10: Percentage Points for  $W'$  Test Statistic  
 Reproduced from Table 1 of Shapiro and Francia [31].

| $n$ | $P$   |       |       |       |       |       |       |       |       |       |       |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
|     | 0.01  | 0.05  | 0.10  | 0.15  | 0.20  | 0.50  | 0.80  | 0.85  | 0.90  | 0.95  | 0.99  |
| 35  | 0.919 | 0.943 | 0.952 | 0.956 | 0.964 | 0.976 | 0.982 | 0.985 | 0.987 | 0.989 | 0.992 |
| 50  | .935  | .953  | .963  | .968  | .971  | .981  | .987  | .988  | .990  | .991  | .994  |
| 51  | 0.935 | 0.954 | 0.964 | 0.968 | 0.971 | 0.981 | 0.988 | 0.989 | 0.990 | 0.992 | 0.994 |
| 53  | .938  | .957  | .964  | .969  | .972  | .982  | .988  | .989  | .990  | .992  | .994  |
| 55  | .940  | .958  | .965  | .971  | .973  | .983  | .988  | .990  | .991  | .992  | .994  |
| 57  | .944  | .961  | .966  | .971  | .974  | .983  | .989  | .990  | .991  | .992  | .994  |
| 59  | .945  | .962  | .967  | .972  | .975  | .983  | .989  | .990  | .991  | .992  | .994  |
| 61  | 0.947 | 0.963 | 0.968 | 0.973 | 0.975 | 0.984 | 0.990 | 0.990 | 0.991 | 0.992 | 0.994 |
| 63  | .947  | .964  | .970  | .973  | .976  | .984  | .990  | .991  | .992  | .993  | .994  |
| 65  | .948  | .965  | .971  | .974  | .976  | .985  | .990  | .991  | .992  | .993  | .995  |
| 67  | .950  | .966  | .971  | .974  | .977  | .985  | .990  | .991  | .992  | .993  | .995  |
| 69  | .951  | .966  | .972  | .976  | .978  | .986  | .990  | .991  | .992  | .993  | .995  |
| 71  | 0.953 | 0.967 | 0.972 | 0.976 | 0.978 | 0.986 | 0.990 | 0.991 | 0.992 | 0.994 | 0.995 |
| 73  | .956  | .968  | .973  | .976  | .979  | .986  | .991  | .992  | .993  | .994  | .995  |
| 75  | .956  | .969  | .973  | .976  | .979  | .986  | .991  | .992  | .993  | .994  | .995  |
| 77  | .957  | .969  | .974  | .977  | .980  | .987  | .991  | .992  | .993  | .994  | .996  |
| 79  | .957  | .970  | .975  | .978  | .980  | .987  | .991  | .992  | .993  | .994  | .996  |
| 81  | 0.958 | 0.970 | 0.975 | 0.979 | 0.981 | 0.987 | 0.992 | 0.992 | 0.993 | 0.994 | 0.996 |
| 83  | .960  | .971  | .976  | .979  | .981  | .988  | .992  | .992  | .993  | .994  | .996  |
| 85  | .961  | .972  | .977  | .980  | .981  | .988  | .992  | .992  | .993  | .994  | .996  |
| 87  | .961  | .972  | .977  | .980  | .982  | .988  | .992  | .992  | .993  | .994  | .996  |
| 89  | .961  | .972  | .977  | .981  | .982  | .988  | .992  | .993  | .994  | .995  | .996  |
| 91  | 0.962 | 0.973 | 0.978 | 0.981 | 0.983 | 0.989 | 0.992 | 0.993 | 0.994 | 0.995 | 0.996 |
| 93  | .963  | .973  | .979  | .981  | .983  | .989  | .992  | .993  | .994  | .995  | .996  |
| 95  | .965  | .974  | .979  | .981  | .983  | .989  | .993  | .993  | .994  | .995  | .996  |
| 97  | .965  | .975  | .979  | .982  | .984  | .989  | .993  | .993  | .994  | .995  | .996  |
| 99  | .967  | .976  | .980  | .982  | .984  | .989  | .993  | .994  | .994  | .995  | .996  |